Cambium seminar

Coq Coq Correct!

Verification of Type Checking and Erasure for Coq, in Coq

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joint work with

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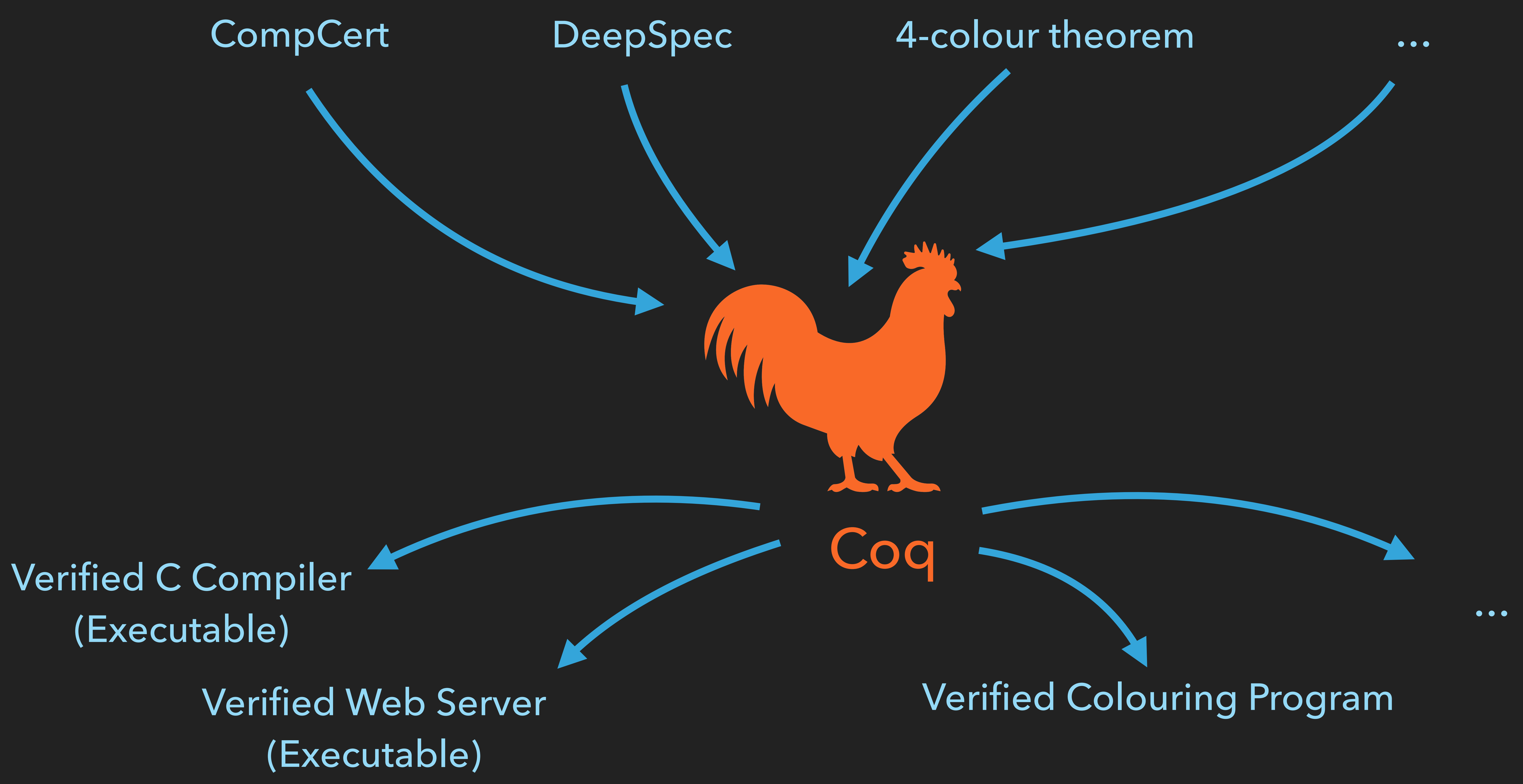
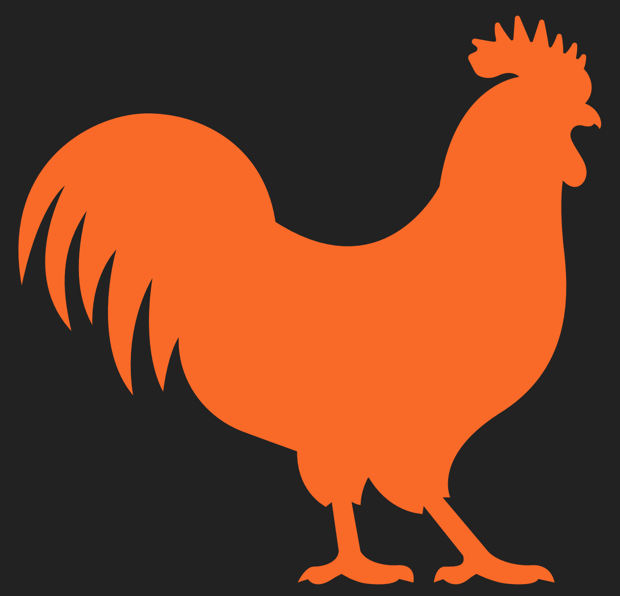
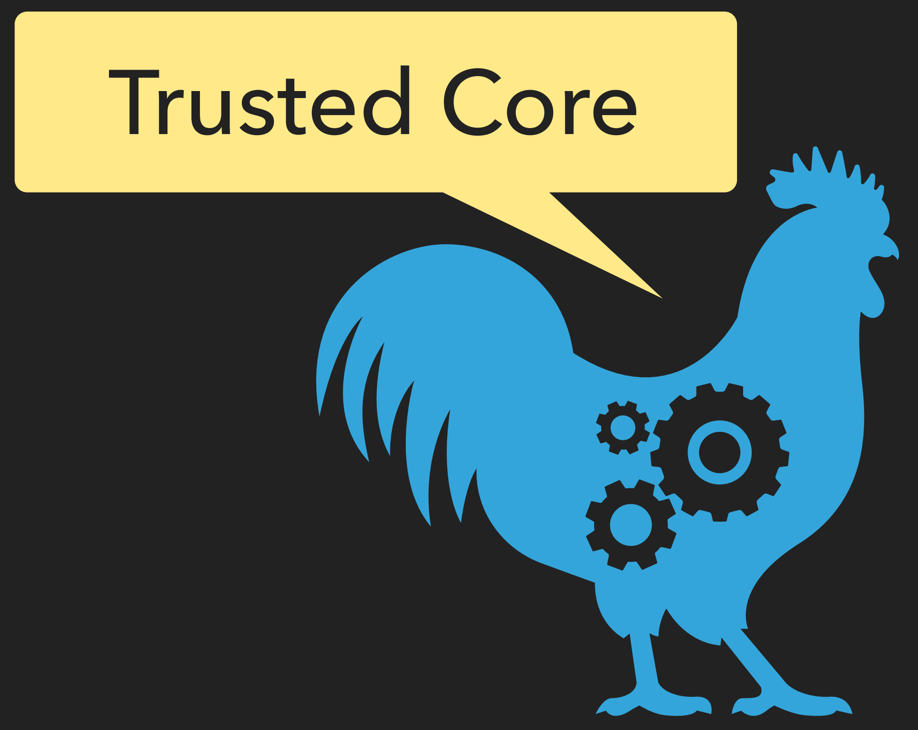
Motivation


CompCert
DeepSpec
4-colour theorem
...
Coq
Verified C Compiler
(Executable)
Verified Web Server
(Executable)
Verified Colouring Program
...

What do you trust?



Ideal Coq

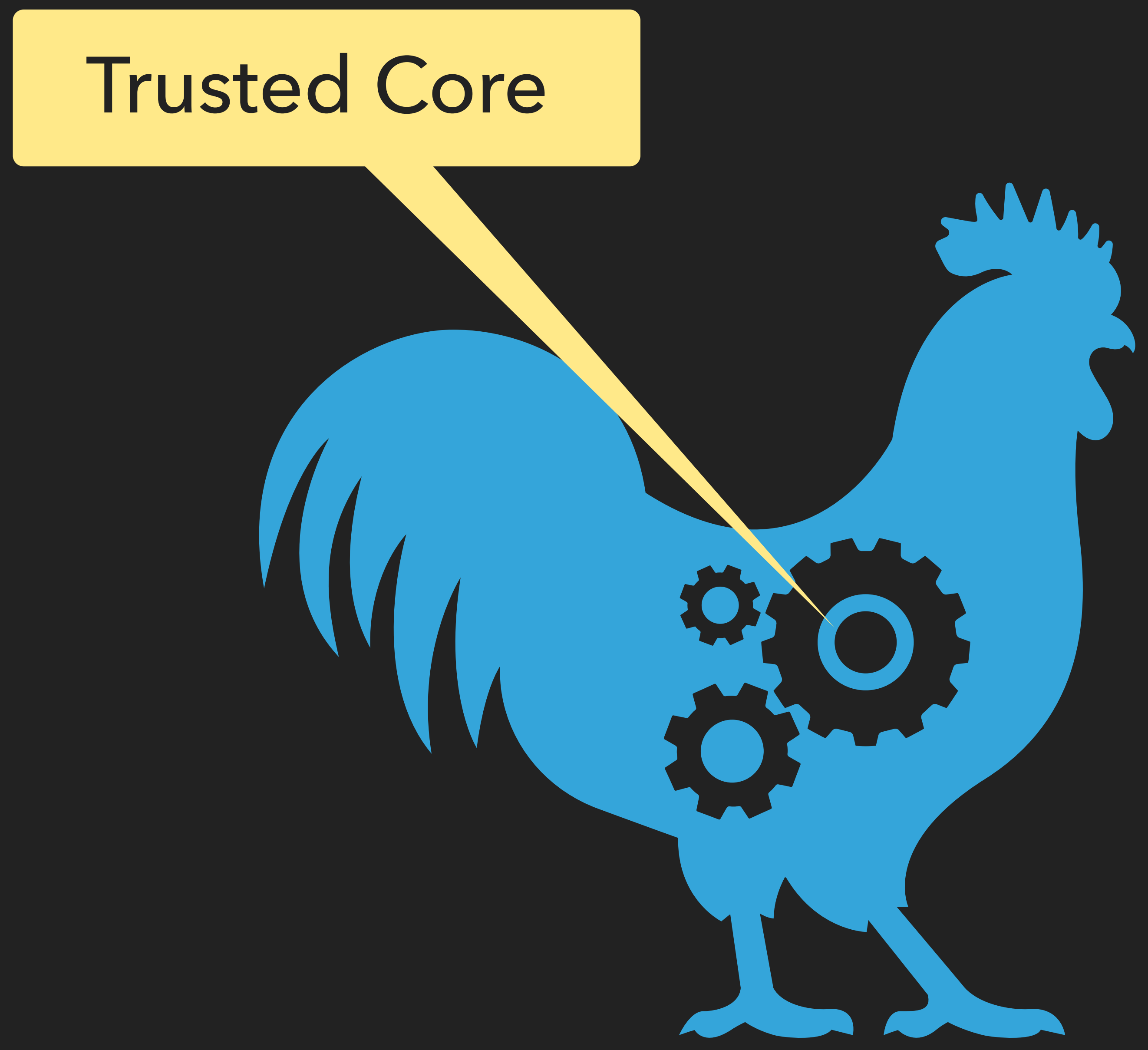


Implemented Coq

What do you trust?

Dependent Type Checker (18kLoC, 30+ years)

- Inductive Families w/ Guard Checking
- Universe Cumulativity and Polymorphism
- ML-style Module System
- KAM, VM and Native Conversion Checkers
- OCaml's Compiler and Runtime



Implemented Coq

The Reality

Unspecified

Ideal Coq

Buggy

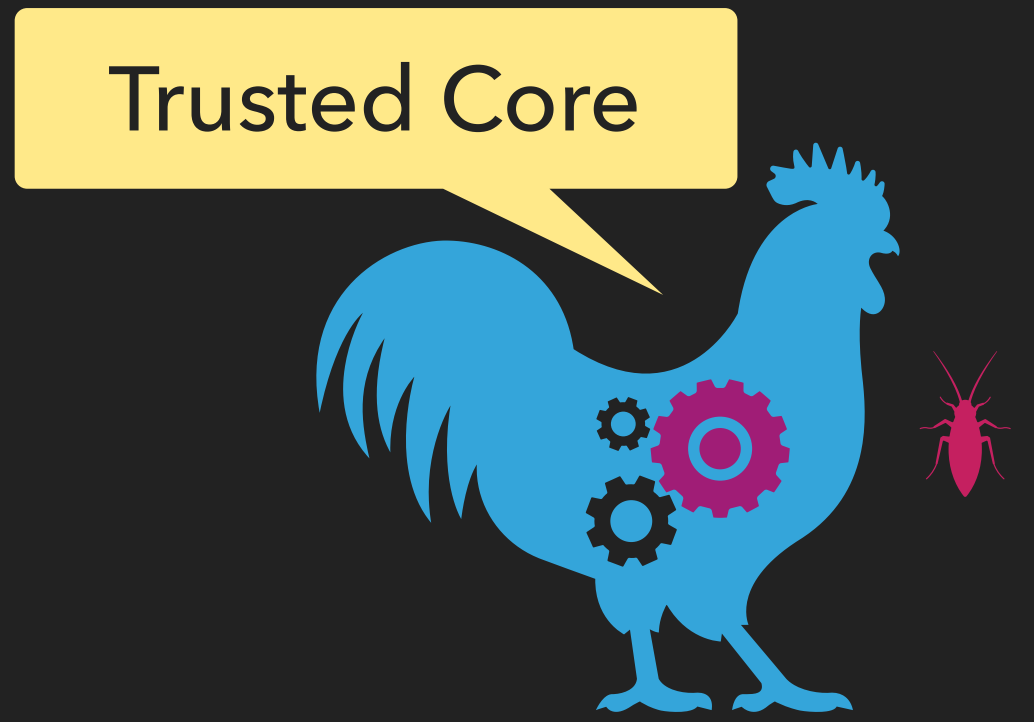
Implemented Coq

The Reality

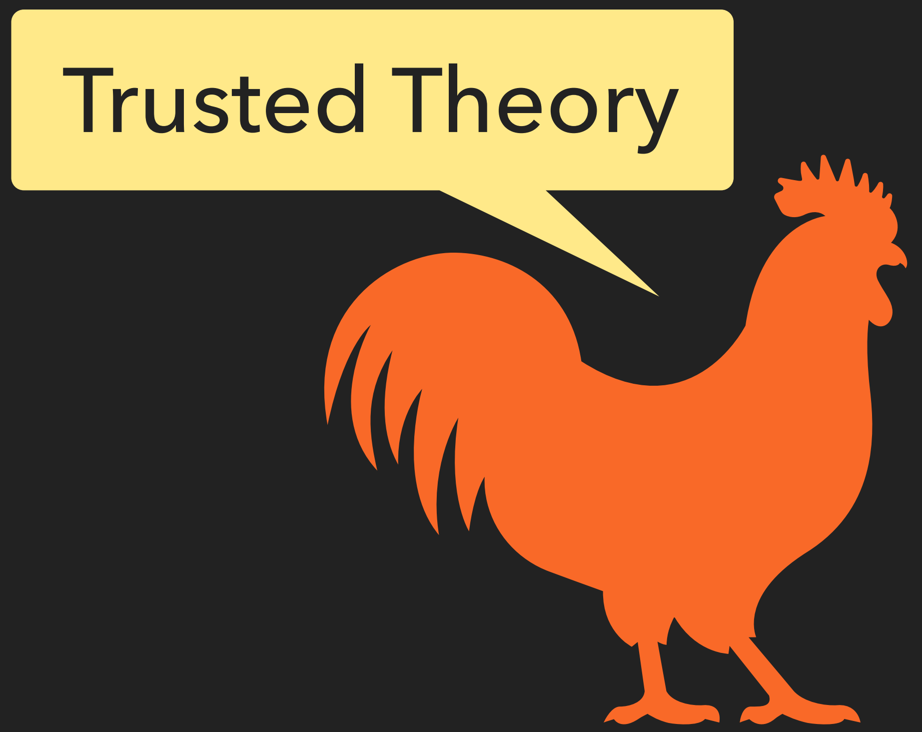
- Reference Manual roughly specifies on paper the basic core metatheory. The rest is (at best) in various papers and PhD theses, e.g. module system, treatment of eta-conversion, guard condition, SProp….
- Discrepancies with the actual implementation
- Combination of features not worked-out in detail. E.g. cumulative inductive types + let-bindings in parameters of inductives???

The Reality

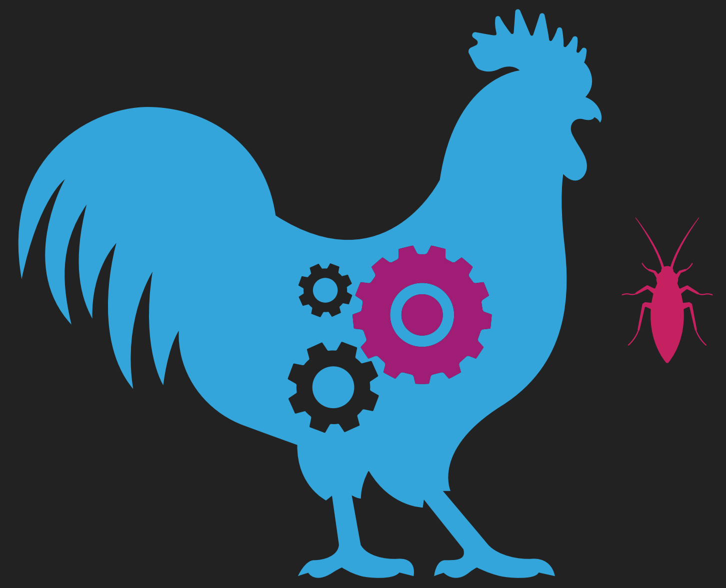
```
354 lines (314 sloc)  16.7 KB

Preliminary compilation of critical bugs in stable releases of Coq
==================================================================
  WORK IN PROGRESS WITH SEVERAL OPEN QUESTIONS


To add: #7723 (vm_compute universe polymorphism), #7695 (modules an

Typing constructions

  component: "match"
  summary: substitution missing in the body of a let
  introduced: ?
  impacted released versions: V8.3-V8.3pl2, V8.4-V8.4pl4
  impacted development branches: none
  impacted coqchk versions: ?
  fixed in: master/trunk/v8.5 (e583a79b5, 22 Nov 2015, Herbelin), v
  found by: Herbelin
```



Our Goal: Improving Trust



Ideal Coq



~ 1 critical bug every year

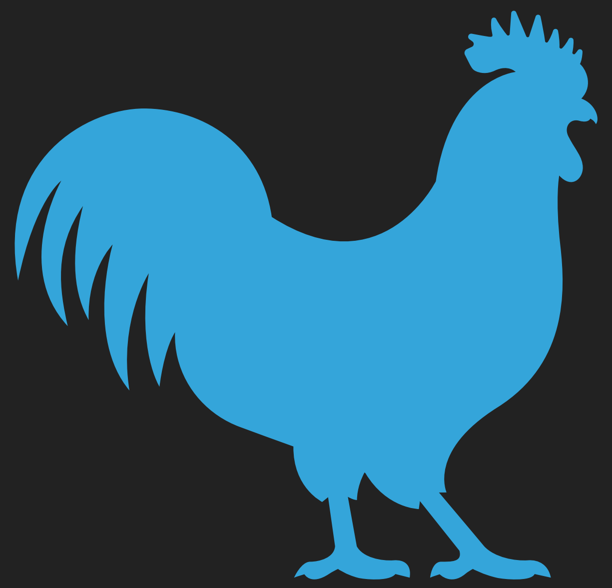
Implemented Coq

Coq in MetaCoq

Trusted Theory

Part I: Coq's Calculus PCUIC

↕

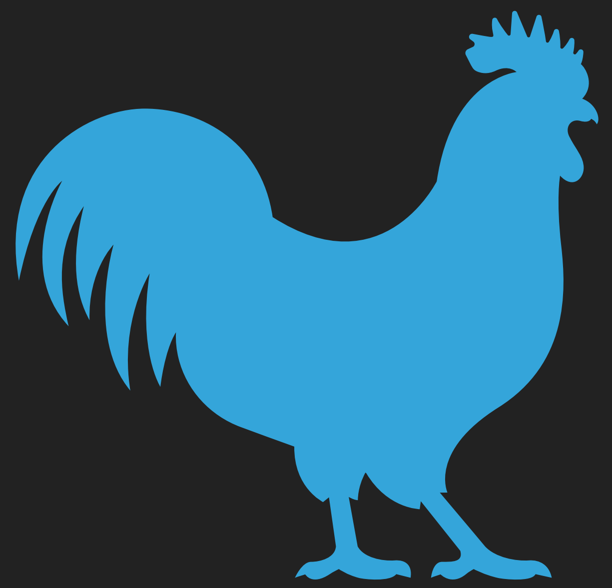


Part II: Verified Coq

in

MetaCoq
Formalization of
Coq in Coq
JAR'20

in



Implemented Coq

What we have…

```
fix vrev {A : Type@{i}} {n m : nat} (v : vec@{i} A n) (acc : vec@{i} A m) :=
  match v in vec _ n return vec@{i} A (n + m) with
  | vnil          ⇒ acc
  | vcons a n v’ ⇒
      let idx := S n + m in
      coerce (vec A) idx (e : n + S m = idx) (vrev v’ (vcons a m acc))
  end.
```

```
vrev_term : term :=
tFix [{|
  dname := nNamed "vrev" ;
  dtype := tProd (nNamed « A") (tSort (Universe.make'' (Level.Level "Top.160", false) []))
    (tProd (nNamed "n") (tInd {| inductive_mind := "Coq.Init.Datatypes.nat";
      inductive_ind := 0 |} [])
    (tProd (nNamed "m") (tInd {| ...
```

What we have…

```
fix vrev {A : Type@{i}} {n m : nat} (v : vec@{i} A n) (acc : vec@{i} A m) :=
  match v in vec _ n return vec@{i} A (n + m) with
  | vnil          ⇒ acc
  | vcons a n v’ ⇒
      let idx := S n + m in
      coerce (vec A) idx (e : n + S m = idx) (vrev v’ (vcons a m acc))
  end.
```

...and what we don't

```
(fun x ⇒ f x) ≡ f   (x ∉ f)
```

η-conversion (WIP)

```
list nat : Set
list Type@{i} : Type@{i}
```

« template » polymorphism

```
Module M <: S. Definition t := nat. End M.
```

module system

No existential or named variables (yet)

Specification

Example: Reduction

DEFINITIONS IN CONTEXTS

$$\frac{(x : T := t) \in \Gamma}{\Gamma \vdash x \to t}$$

GENERAL SUBSTITUTION

$$\frac{}{\Gamma \vdash \mathtt{let}\ x : T := t\ \mathtt{in}\ b \to b'[x := t]}$$

STRONG REDUCTION

$$\frac{\Gamma, x : T := t \vdash b \to b'}{\Gamma \vdash \mathtt{let}\ x : T := t\ \mathtt{in}\ b \to \mathtt{let}\ x : T := t\ \mathtt{in}\ b'}$$

Specification

Example: Call-by-Value Evaluation

WEAK REDUCTION

CLOSED VALUE SUBSTITUTION

$$\frac{t \to_{cbv} v \qquad b[x := v] \to_{cbv} v'}{\texttt{let } x : T := t \texttt{ in } b \to_{cbv} v'}$$

$$\_ \to_{cbv} \_ \;\subseteq\; \varepsilon \vdash \_ \to \_$$

Meta-Theory

Structures

```
term, t, u ::=
  | Rel (n : nat) | Sort (u : universe) | …

global_env, Σ ::= []
 | Σ , (kername × InductiveDecl idecl)
 | Σ , (kername × ConstantDecl cdecl)

global_env_ext ::= (global_env × universes_decl)

Γ ::= []
   | Γ , aname : term
   | Γ , aname := t : u
```

(global environment)

(global environment with universes)

(local environment)

Meta-Theory

Judgments

$\Sigma \; ; \; \Gamma \vdash t \to u, \; t \to^* u$

One-step reduction and its reflexive transitive closure

$\Sigma \; ; \; \Gamma \vdash t =_\alpha u, \; t \leq_\alpha u$

α-equivalence + equality or cumulativity of universes

$\Sigma \; ; \; \Gamma \vdash T = U, \; T \leq U$

Conversion and cumulativity

$\iff T \to^* T' \wedge U \to^* U' \wedge T' \leq_\alpha U'$

$\Sigma \; ; \; \Gamma \vdash t : T$

Typing

`wf Σ, wf_local Σ Γ`

Well-formed global and local environments

Basic Meta-Theory

Structural Properties

- Traditional de Bruijn lifting and substitution operations in the spec
- Show that σ-calculus operations simulate them (à la Autosubst) :

```
ren : (nat -> nat) -> term -> term
inst : (nat -> term) -> term -> term
```

- **Weakening** and **Substitution** from renaming and instantiation theorems
- Easier to lift to strengthening/exchange lemmas in the future (strengthening is not immediate here)

Universes

```
universe ::= Prop | SProp
  | Type (ne_sorted_list (universe_level * nat)).
```

Typing $\Sigma \, ; \, \Gamma \vdash$ `tSort u` $:$ `tSort (Universe.super u)`

No distinction of *algebraic* universes (more general than current Coq)

```
universe_constraint ::=
  universe_level × ℤ × universe_level.      (u + x ≤ v)
```

Specification Global set of consistent constraints, satisfy a valuation in $\mathbb{N}$.

- `lSet` always has level `0`, smaller than any other universe.
- Impredicative sorts are separate from the predicative hierarchy.

Universes

Basic Meta-Theory

Global environment weakening

Monotonicity of typing under context extension: universe consistency is monotone.

Universe instantiation

Easy, de Bruijn level encoding of universe variables (no capture)

Implementation

Longest simple paths in the graph generated by the constraints `ϕ`, with source `lSet`

```
∀ l, lsp ϕ l l = 0 ⟺ satisfiable ϕ (λ l, lsp lSet l)
```

Meta-Theory

The path to subject reduction

Validity

$$\frac{\Sigma \,;\, \Gamma \vdash t : T}{\Sigma \,;\, \Gamma \vdash T : \mathtt{tSort}\ s}$$

Requires **transitivity** of conversion/cumulativity

Context Conversion

$$\frac{\Sigma \,;\, \Gamma \vdash t : T \qquad \Sigma \vdash \Delta \leq \Gamma}{\Sigma \,;\, \Delta \vdash t : T}$$

More generally, context **cumulativity**

Subject Reduction

$$\frac{\Sigma \,;\, \Gamma \vdash t : T \qquad \Sigma \,;\, \Gamma \vdash t \to u}{\Sigma \,;\, \Gamma \vdash u : T}$$

Relies on injectivity of product types, a consequence of **confluence**

Confluence

The traditional way

$\Sigma \; , \; \Gamma \vdash t \Rrightarrow u$ One-step parallel reduction

À la Tait-Martin-Löf/Takahashi:

Diamond for $\Rrightarrow$

$$\begin{tikzcd} & t \arrow[dl, Rrightarrow] \arrow[dr, Rrightarrow] & \\ v \arrow[dr, Rrightarrow] & & u \arrow[dl, Rrightarrow] \\ & \exists\, t' & \end{tikzcd}$$

"Squash" lemma

$$\_ \rightarrow \_ \;\subset\; \_ \Rrightarrow \_ \;\subset\; \_ \rightarrow^{*} \_$$

Takahashi's Trick

```
ρ : term -> term
```

An *optimal* one-step parallel reduction function.

$$t \Rrightarrow \rho(t)$$

The triangle property

$$\begin{tikzcd} t \arrow[d, Rrightarrow] \arrow[dr, Rrightarrow] & \\ \rho(t) & u \arrow[l, Rrightarrow] \end{tikzcd}$$

The triangle property

$$\begin{tikzcd} & t \arrow[dl, Rrightarrow] \arrow[dr, Rrightarrow] \arrow[dd, Rrightarrow] & \\ v \arrow[dr, Rrightarrow] & & u \arrow[dl, Rrightarrow] \\ & \rho(t) & \end{tikzcd}$$

Confluence

For a theory with definitions in contexts

$$\Sigma \vdash \Gamma,\ t \Rrightarrow \Delta,\ u$$

One-step parallel reduction, including reduction in contexts.

$$\frac{\Sigma \vdash \Gamma,\ x := t \Rrightarrow \Delta,\ x := t' \qquad \Sigma \vdash (\Gamma,\ x := t),\ b \Rrightarrow (\Delta,\ x := t'),\ b'}{\Sigma \vdash \Gamma,\ (\mathtt{let}\ x := t\ \mathtt{in}\ b) \Rrightarrow \Delta,\ (\mathtt{let}\ x := t'\ \mathtt{in}\ b')}$$

```
ρ : context -> term -> term
ρctx : context -> context
```

$$\begin{tikzcd} & \Gamma, t \arrow[dl, Rrightarrow] \arrow[dd, Rrightarrow] \arrow[dr, Rrightarrow] & \\ \Gamma', u \arrow[dr, Rrightarrow] & & \Gamma'', v \arrow[dl, Rrightarrow] \\ & \rho ctx\ \Gamma,\ \rho\ (\rho ctx\ \Gamma)\ t & \end{tikzcd}$$

Principality and changing equals for equals

```
Definition principality {Σ Γ t} : (welltyped Σ Γ t : Prop) →
  ∑ (P : term), Σ ; Γ ⊢ t : P × principal_type Σ Γ t P
```

$$\frac{\Sigma\ ;\ \Gamma \vdash t : T \qquad \Sigma\ ;\ \Gamma \vdash u : U \qquad \Sigma \vdash u \leq_{\alpha\_noind} t}{\Sigma\ ;\ \Gamma \vdash u : T}$$

Informally: (well-typed) smaller terms have more types than larger ones.

Justifies the `change` tactic up-to cumulativity (excluding inductive type cumulativity).

Cumulativity and Prop

$\Sigma \; ; \; \Gamma \vdash T \sim U$

Conversion identifying all predicative universes (hence larger than cumulativity).

$$\frac{\Sigma \; ; \; \Gamma \vdash t : T \qquad \Sigma \; ; \; \Gamma \vdash u : U \qquad \Sigma \vdash u \leq_\alpha t}{\Sigma \; ; \; \Gamma \vdash T \sim U}$$

Informally: for two well-typed terms, if they are syntactically equal up-to cumulativity of inductive types, then they live in the same hierarchy (`Prop`, `SProp` or `Type`)

Required for erasure correctness

Trusted Theory Base

Assumptions

- The specifications of typing, reduction and cumulativity
  ~ 500 LoC from scratch (verified and testable)

- Guard Conditions. Oracles:
  `check_fix : global_env → context → fixpoint → bool`
  \+ preservation by renaming/instantiation/equality/reduction

- Strong Normalization (not provable thanks to Gödel, but also not used in the preceding results). Consistency and canonicity follow easily.

```
Axiom normalisation :
  ∀ Γ t, welltyped Σ Γ t → Acc (cored (fst Σ) Γ) t.
```

Verifying Type-Checking

Conversion

Objective

Input

$u : A$ $v : B$

Output

$(u \equiv v) + (u \not\equiv v)$

Conversion

Objective

Input

`u : A` `v : B`

Output

`(u ≡ v) + (u ≢ v)`

```
isconv :
  ∀ Σ Γ (u v A B : term),
    (Σ ; Γ ⊢ u : A) ?
    (Σ ; Γ ⊢ v : B) ?
    (Σ ; Γ ⊢ u ≡ v) +
    (Σ ; Γ ⊢ u ≡ v ? ⊥)
```

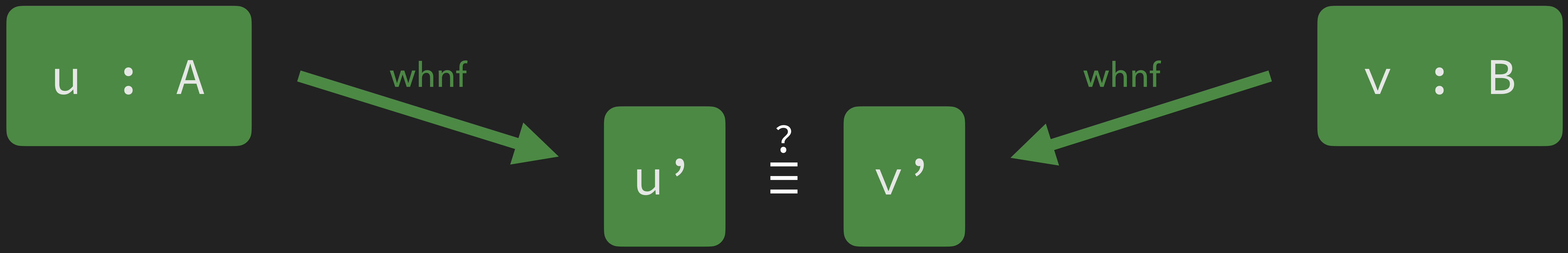
Conversion

Algorithm

$u : A \xrightarrow{\text{whnf}} u'$ $\qquad$ $v' \xleftarrow{\text{whnf}} v : B$

Conversion

Algorithm



Conversion

Algorithm

$$u : A \xrightarrow{\text{whnf}} u' \stackrel{?}{\equiv} v' \xleftarrow{\text{whnf}} v : B$$

match u' , v' **with**

$\lambda(x{:}A_1).\ t_1$, $\lambda(x{:}A_2).\ t_2$ $\Rightarrow$ $A_1 \stackrel{?}{\equiv} A_2 \ \wedge\ t_1 \stackrel{?}{\equiv} t_2$

$\Pi(x{:}A_1).\ B_1$, $\Pi(x{:}A_2).\ B_2$ $\Rightarrow$ $A_1 \stackrel{?}{\equiv} A_2 \ \wedge\ B_1 \stackrel{?}{\equiv} B_2$

Conversion

Completeness

$u : A \xrightarrow{\text{whnf}} u'$ $\qquad$ $v' \xleftarrow{\text{whnf}} v : B$

$\mathtt{match}\ u',\ v'\ \mathtt{with}$

$\lambda(x{:}A_1).\ t_1\ ,\ \lambda(x{:}A_2).\ t_2 \Rightarrow A_1 \overset{?}{\equiv} A_2 \wedge t_1 \overset{?}{\equiv} t_2$

$\Pi(x{:}A_1).\ B_1\ ,\ \Pi(x{:}A_2).\ B_2 \Rightarrow A_1 \overset{?}{\equiv} A_2 \wedge B_1 \overset{?}{\equiv} B_2$

Conversion

Completeness

$$\Pi(x{:}A_1).\ B_1 \overset{?}{\equiv} \Pi(x{:}A_2).\ B_2 \Rightarrow A_1 \not\equiv A_2$$

Conversion

Completeness

$\Pi(x{:}A_1).\ B_1 \stackrel{?}{\equiv} \Pi(x{:}A_2).\ B_2 \Rightarrow A_1 \not\equiv A_2$

we conclude

$\Pi(x{:}A_1).\ B_1 \not\equiv \Pi(x{:}A_2).\ B_2$

using inversion lemmata and confluence

Conversion

u : A $\xrightarrow{\text{whnf}}$ u'   v' $\xleftarrow{\text{whnf}}$ v : B

```
match u' , v' with
λ(x:A₁). t₁ , λ(x:A₂). t₂ ⇒ A₁ ≟ A₂ ∧ t₁ ≟ t₂
Π(x:A₁). B₁ , Π(x:A₂). B₂ ⇒ A₁ ≟ A₂ ∧ B₁ ≟ B₂
```

Weak head reduction

Objective

Input	Output
u	v
`term`	`term`

Weak head reduction

Objective

Input	Output	
u	v	u $\boxed{?}$ v
`term`	`term`	`Prop`

Weak head reduction

Objective

Input

u

`term`

Output

v

`term`

u ⍰ v

`Prop`

```
weak_head_reduce : ∀ (u : term), ∑ (v : term), u ⍰ v
```

Weak head reduction

Example

Input u

Output v u ?

```
Definition foo := λ(x:nat). x.
```

```
foo 0
```

Weak head reduction

Example

Input u

Output v u ?

```
Definition foo := λ(x:nat). x.
```

foo 0

foo ⟶ λ(x:nat).x

Weak head reduction

Example

Input u

Output v u ?

```
Definition foo := λ(x:nat). x.
```

λ(x:nat).x 0

foo ⟶ λ(x:nat).x

Weak head reduction

Example

Input u

Output v u ?

```
Definition foo := λ(x:nat). x.
```

0

foo ⟶ λ(x:nat).x

Weak head reduction

Example

Input u

Output v | u ?

```
Definition foo := λ(x:nat). x.
```

0

foo 0 ⟶ (λ(x:nat).x) 0 ⟶ 0

Weak head reduction

Termination

Input

u

Output

v

u ? v

Weak head reduction

Termination

Input

$u \quad \pi_1$

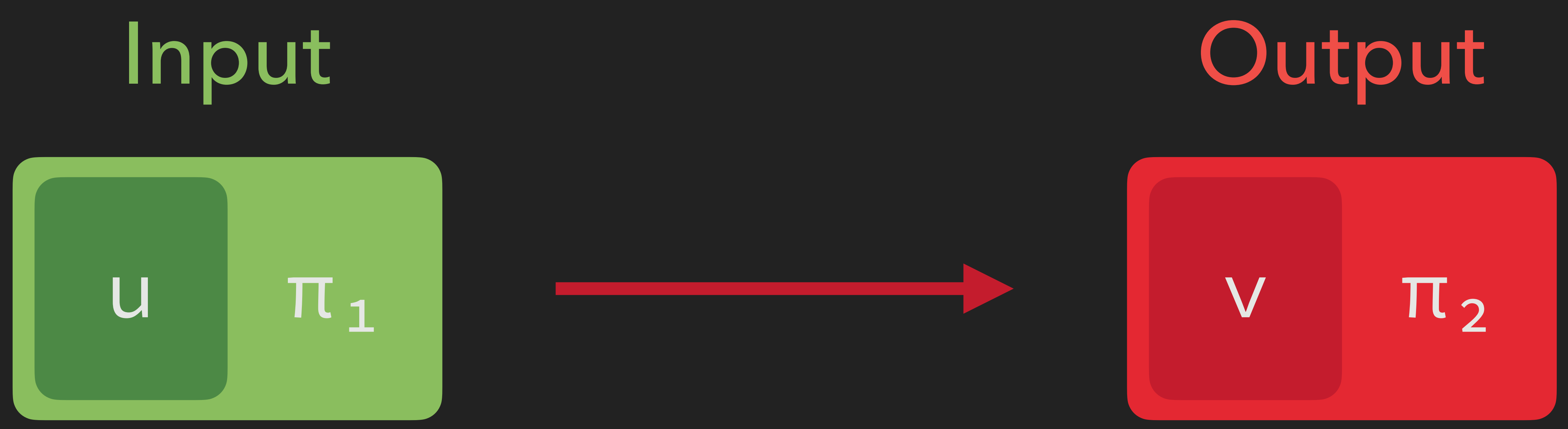
Output

$v \quad \pi_2$

$u \; \boxed{?} \; v$

Weak head reduction

Termination



Weak head reduction

Termination

`foo 0` ⟶ `foo 0` ⟶ `λ(x:nat).x 0` ⟶ `0`

Weak head reduction

Termination

`foo 0` `foo 0` `λ(x:nat).x 0` `0`

`(λ(x:nat).x) 0` ⟶ `0`

Weak head reduction

Termination

```
foo 0 ⟶ (λ(x:nat).x) 0
```

```
foo 0    foo 0    λ(x:nat).x 0    0
```

```
(λ(x:nat).x) 0 ⟶ 0
```

Weak head reduction

Termination

foo 0 ⟶ (λ(x:nat).x) 0

foo 0 | foo 0 | λ(x:nat).x 0 | 0

foo 0 ⊐ foo

(λ(x:nat).x) 0 ⟶ 0

Weak head reduction

Termination

$\texttt{foo 0} \longrightarrow \texttt{(λ(x:nat).x) 0}$

foo 0	foo 0	λ(x:nat).x 0	0

$\texttt{foo 0} \sqsupset \texttt{foo}$

$\texttt{(λ(x:nat).x) 0} \longrightarrow \texttt{0}$

Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

`foo 0` ⟶ `(λ(x:nat).x) 0`

`foo 0` | `foo` `0` | `λ(x:nat).x` `0` | `0`

`foo 0 ⊐ foo`

`and foo 0 = foo 0`

`(λ(x:nat).x) 0` ⟶ `0`

1

Lexicographic order of $\boxed{?}$ and ⊏

Weak head reduction

Termination

`p.1`

1 Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

p.1

1 Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

$\boxed{\texttt{p.1}}$ $\quad$ $\boxed{\texttt{p}}\texttt{.1}$

but `p.1` $\neq$ `p`

1 Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

`p.1` > `p.1`

`and p.1 = p.1`

1 Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

```
fix f (n:nat). t end n
```

Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

```
fix f (n:nat). t end n
```

1 Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

```
fix f (n:nat). t end n
```

1 Lexicographic order of $\boxed{?}$ and $\sqsubset$

Weak head reduction

Termination

```
fix f (n:nat). t end n
```

▼

```
fix f (n:nat). t end n
```

~~Lexicographic order of ? and ⊏~~

Weak head reduction

Termination

~~Lexicographic order of ⍰ and ⊏~~

Weak head reduction

Termination

Lexicographic order of $\boxed{?}$ and an order on positions

Weak head reduction

Termination

Lexicographic order of $\boxed{?}$ and an order on positions

Weak head reduction

Termination

$$u \; \pi_1 \;\blacktriangleright\; v \; \pi_2$$

2 Lexicographic order of $\boxed{?}$ and an order on positions

Weak head reduction

Termination

$$u\ \pi_1 \ \blacktriangleright\ v\ \pi_2$$

$$\langle\ u\ \pi_1\ ,\ \underbrace{\texttt{stack\_pos}\ u\ \pi_1}_{\texttt{pos}\ (u\ \pi_1)}\ \rangle > \langle\ v\ \pi_2\ ,\ \underbrace{\texttt{stack\_pos}\ v\ \pi_2}_{\texttt{pos}\ (v\ \pi_2)}\ \rangle$$

2 Lexicographic order of $\boxed{?}$ and an order on positions

Weak head reduction

Termination

$$u \ \pi_1 \ \triangleright \ v \ \pi_2$$

$$\langle \ u \ \pi_1 \ , \ \underbrace{\texttt{stack\_pos} \ u \ \pi_1}_{\texttt{pos} \ (u \ \pi_1)} \ \rangle > \langle \ v \ \pi_2 \ , \ \underbrace{\texttt{stack\_pos} \ v \ \pi_2}_{\texttt{pos} \ (v \ \pi_2)} \ \rangle$$

Dependent lexicographic order of $\boxed{?}$ and an order on positions

Type Checking

Weak head reduction

Conversion

Type Checking

Weak head reduction

Cumulativity

Inference

Type Checking

Weak head reduction

Cumulativity

Inference

`Infer t` → `Check B ≤ A`

`Check t : A`

Type Checking

Weak head reduction

Cumulativity

Inference

`Infer t : B` → `Check B ≤ A`

`Check t : A` ✓

```
MetaCoq Check foo.
```

ITP’21 Bidirectional type checking for completeness

A little success story

Spec/Proof/Program co-design for the new `match` representation in Coq (PR #13563 by P.M. Pédrot, CEP #34 by H. Herbelin).

- MetaCoq => typechecking of case on cumulative inductive types is incomplete
- Failure of subject reduction in Coq.
- "Quick" fix requires strengthening which in turn is not provable without subject reduction, leading to a messy meta-theory. Also incompatible with eta-conversion.
- The new representation solves all these issues **and** reflects the high-level user syntax more faithfully. It's win/win/win!

Benefits of Bidirectional Type-Checking

- MetaCoq => typechecking of case on cumulative inductive types is incomplete
- Failure of subject reduction in Coq.
- “Quick” fix requires strengthening which in turn is not provable without subject reduction, leading to a messy meta-theory. Also incompatible with eta-conversion.
- The new representation solves all these issues **and** reflects the high-level user syntax more faithfully. It’s win/win/win!

Verifying Erasure

Erasure

At the core of the **extraction** mechanism:

$\mathcal{E} : \text{term} \to \Lambda_{\square,\text{match},\text{fix},\text{cofix}}$

Erases non-computational content:

- Type erasure:

  ```
  ℰ (t : Type) = ☐
  ```

- Proof erasure:

  ```
  ℰ (p : P : Prop) = ☐
  ```

```
fix vrev {A : Type@{i}} {n m : nat} (v : vec A n)
(acc : vec A m) :=
  match v in vec _ n return vec A (n + m) with
  | vnil          ⇒ acc
  | vcons a n v’ ⇒
      let idx := S n + m in
      coerce (vec A) idx (e : n + S m = idx)
        (vrev v’ (vcons a m acc))
  end.
```

```
ℰ (vrev) =
```

```
fix vrev n m v acc :=
  match v with
  | vnil          ⇒ acc
  | vcons a n v’ ⇒
      let idx := S n + m in
      coerce ☐ idx ☐ (vrev v’ (vcons a m acc))
  end.
```

Erasure

Singleton elimination principle

Erase propositional content used in computational content:

```
Ɛ (match p in eq _ y with eq_refl ⇒ b end) = Ɛ (b)
```

```
Definition coerce {A} {B : A -> Type) {x} (y : A)
   (e : x = y) : P x -> P y :=
  match e with
  | eq_refl         ⇒ fun p => p
  end.

fix vrev n m v acc :=
  match v with
  | vnil         ⇒ acc
  | vcons a n v’ ⇒
      let idx := S n + m in
      coerce ☐ idx ☐ (vrev v’ (vcons a m acc))
  end.
```

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```
Ɛ (match p in eq _ y with eq_refl ⇒ b end) = Ɛ (b)
```

```
Ɛ (coerce) ~   coerce x y := (fun p ⇒ p)
```

```
Ɛ (vrev) ~   fix vrev n m v acc :=
               match v with
               | vnil          ⇒ acc
               | vcons a n v’ ⇒ vrev v’ (vcons a m acc)
               end.
```

Erasure Correctness

$$\begin{tikzcd} t \arrow[r, phantom, "\to_{cbv}"] \arrow[d, "\mathcal{E}"'] & v \arrow[d, leftrightarrow, dashed, "\text{Observational Equivalence}"] \\ t' \arrow[r, phantom, "\to_{cbv}"] & \exists v' \end{tikzcd}$$

With Canonicity and SN:

$$\begin{array}{ll} & \vdash t : nat \\ \Rightarrow & \vdash t \to n : nat \quad (n \in \mathbb{N}) \\ \Rightarrow & t \to_{cbv} n : nat \\ \Rightarrow & \mathcal{E}\,(t) \to_{cbv} n \end{array}$$

Erasure Correctness

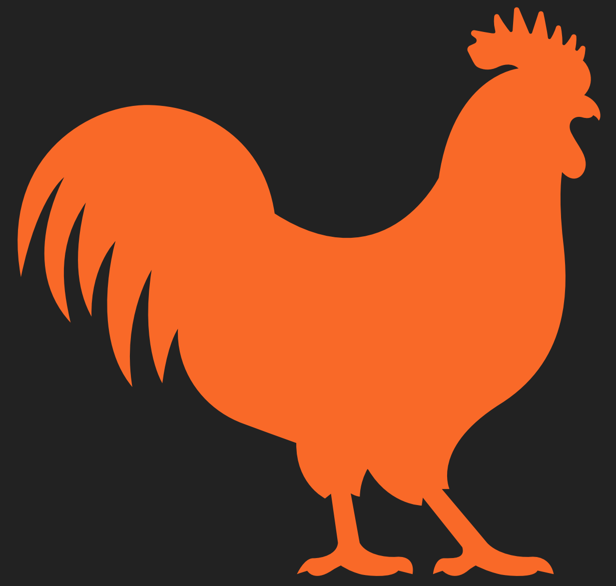
First define a non-deterministic erasure relation, then define:

```
ℰ : ∀ Σ Γ t (wt : welltyped Σ Γ t) → EAst.term
```

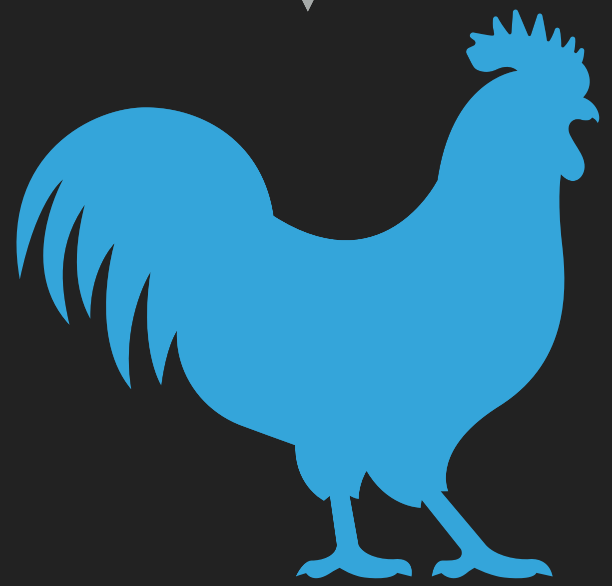
Finally show that ℰ's graph is in the erasure relation. Two additional optimizations:

- Remove trivial cases on singleton inductive types in `Prop`
- Compute the dependencies of the erased term to erase only the computationally relevant subset of the global environment. I.e. remove unnecessary proofs the original term depended on.

Summary



Ideal Coq

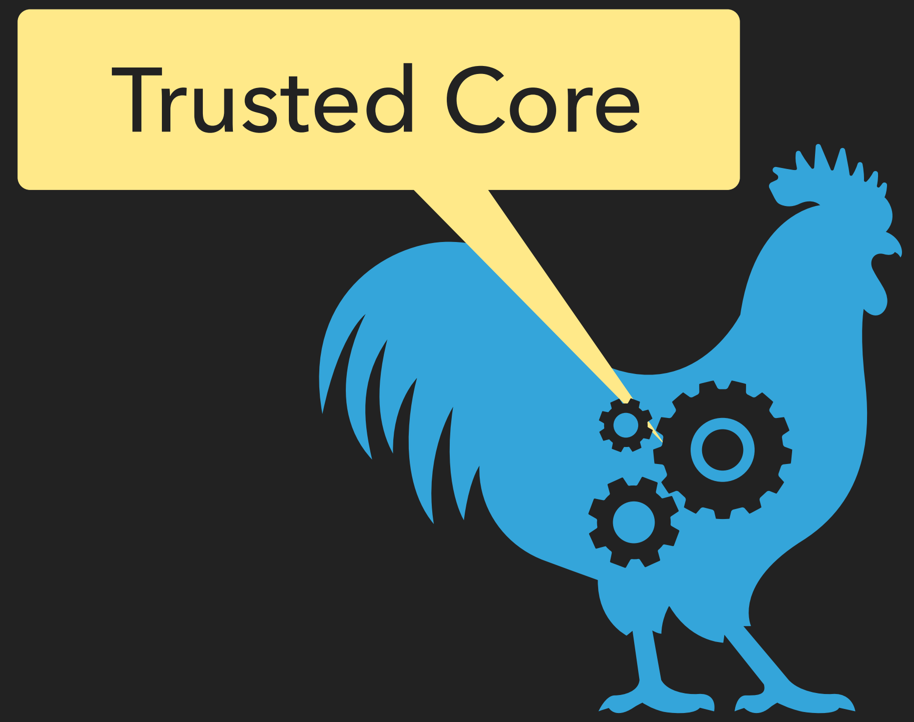


Verified Coq

in

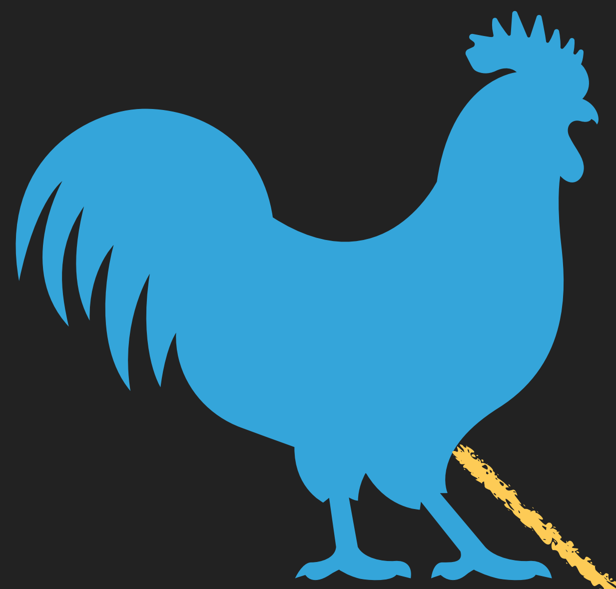
MetaCoq

in



Implemented Coq

Summary



Verified Coq

```
MetaCoq Check vrev.
```

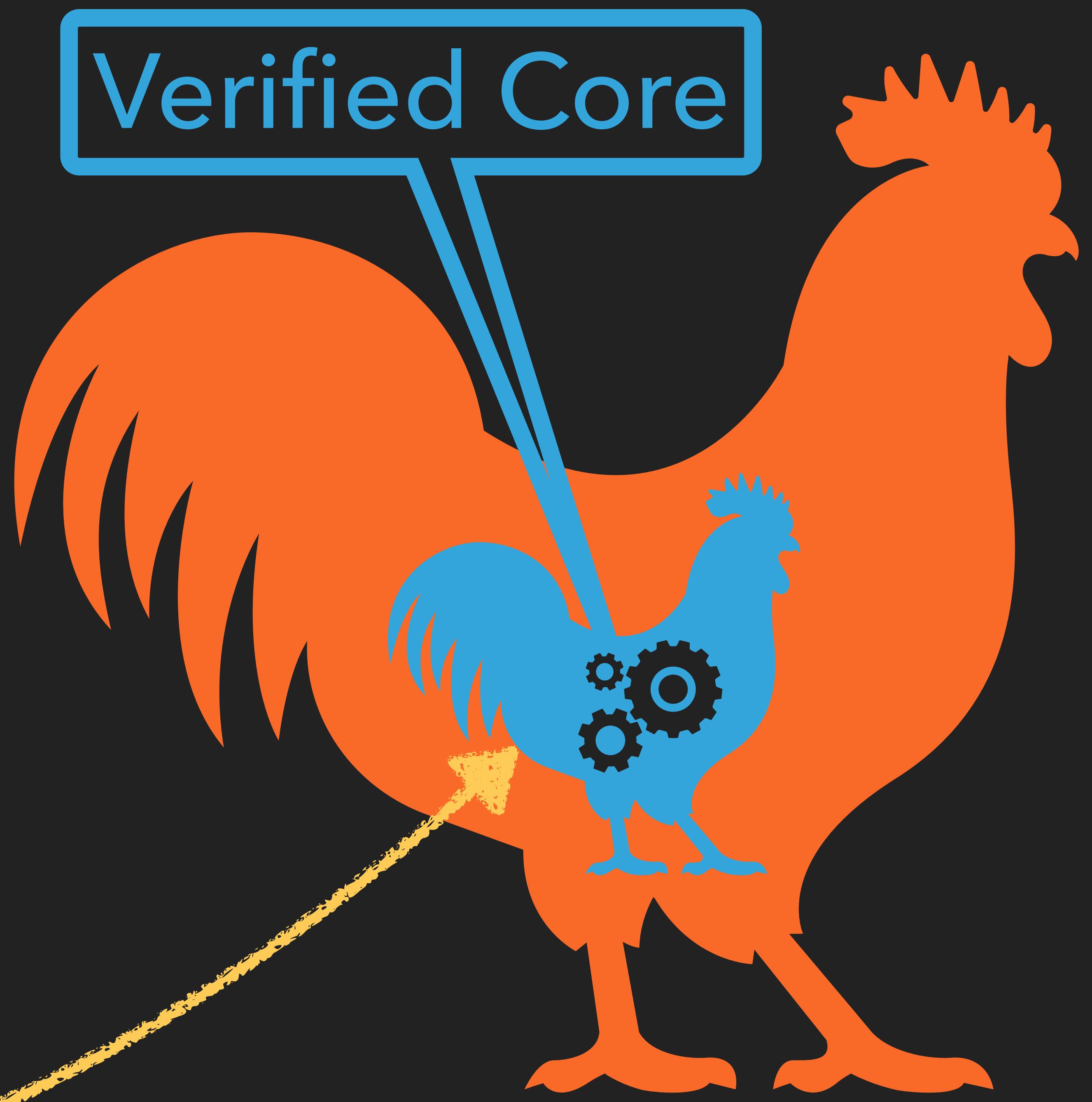
in

MetaCoq

Verified ℰ

```
MetaCoq Erase vrev.
```

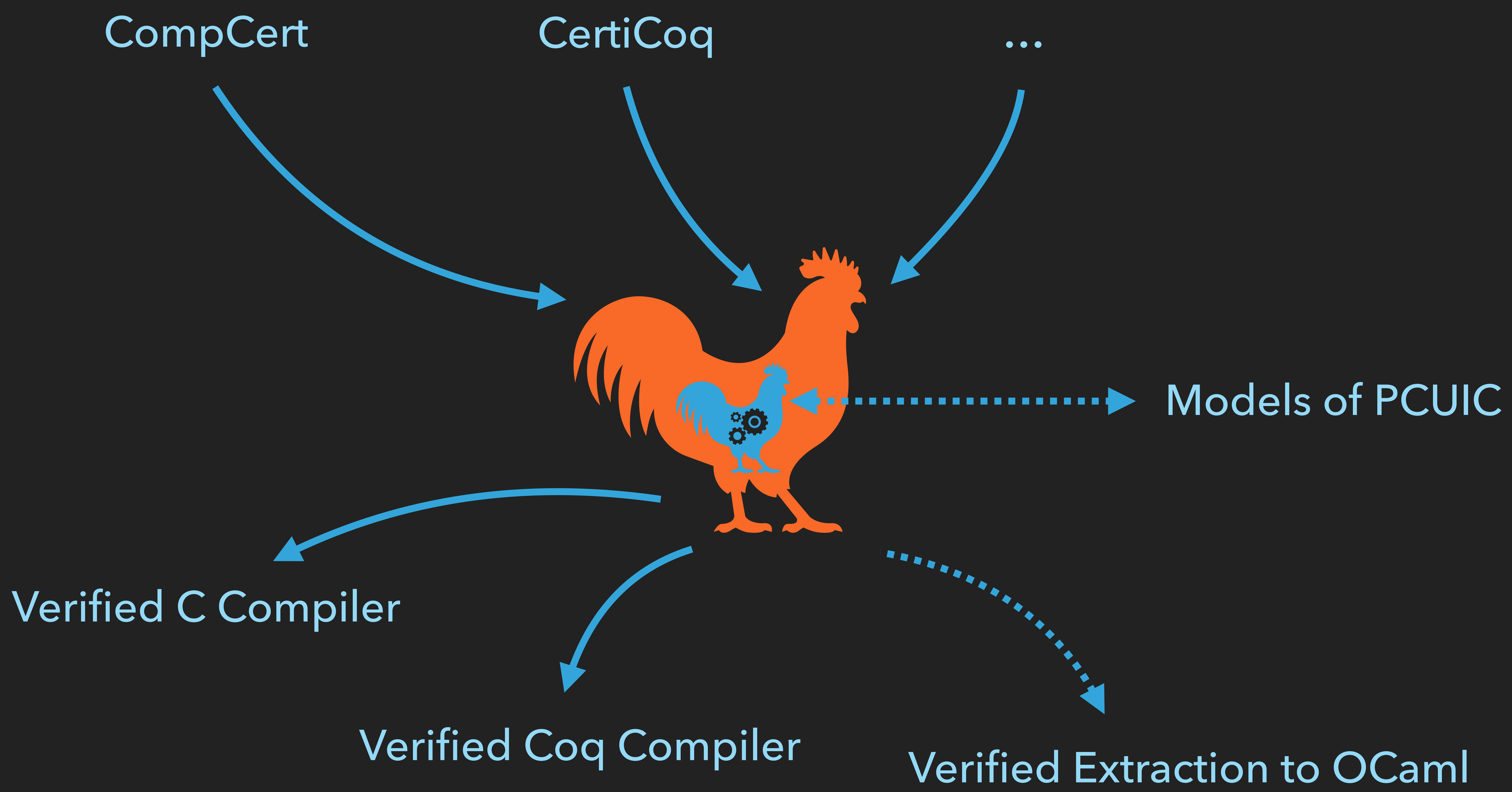
in



Implemented Coq
=
Ideal Coq

Spec:	30kLoC
Proofs:	60kLoC
Comments:	10kLoC

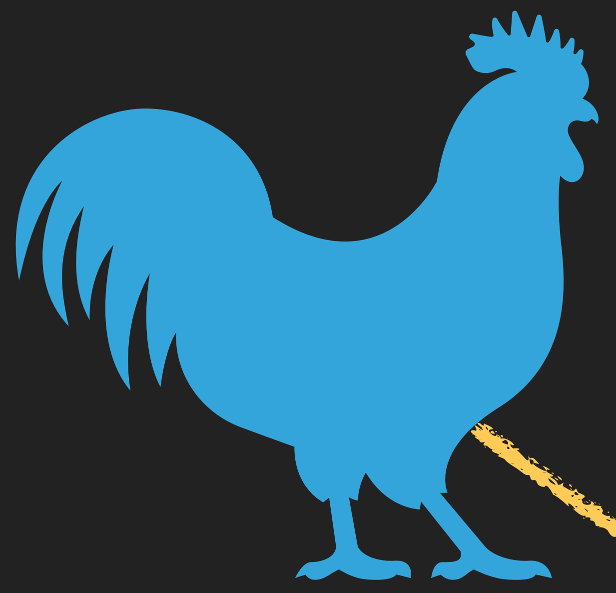
Perspectives



Ongoing and future work

- Integration of **rewrite rules** (CEP #50)
- Interoperability of erased code with OCaml (Nomadic Labs **CoqExtra** project, Pierre Giraud's PhD thesis)
- Full meta-theory for the **SProp** sort and irrelevance checking
- **Eta-reduction** and contravariant subtyping (CEP #47)
- Integration of a **sort-polymorphism** system, generalising universe polymorphism to deal more uniformly with impredicative sorts and alternative hierarchies (exceptional type theory, setoid type theory, erasable sets…) (Kenji Maillard).

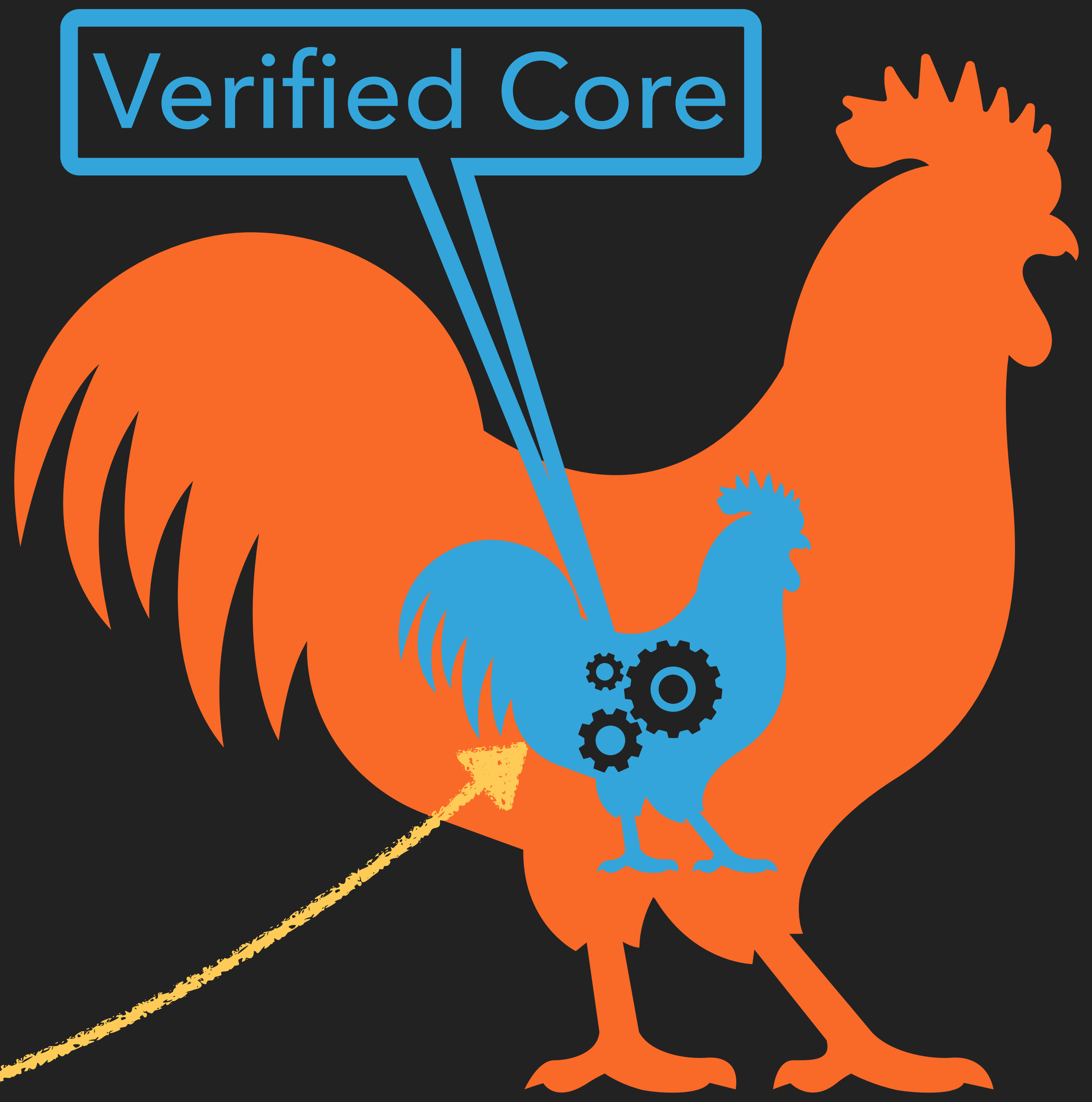
Conclusion



in

MetaCoq

in



Verified Coq

MetaCoq

Verified ℰ

Implemented Coq

=

Ideal Coq

Spec:	30kLoC
Proofs:	60kLoC
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https://metacoq.github.io

Coq in MetaCoq

« *Cot Cot Codet* ». French, Interjection.

1. Cackle (the cry of a hen, especially one that has laid an egg).

Related Work

- Kumar et al., HOL + CakeML (JAR’16)
- Strub et al., Self-Certification of F* starting with Coq (POPL’12)
- Rahli and Anand, NuPRL in Coq (ITP’14)