First-Class Type Classes

Matthieu Sozeau
Joint work with Nicolas Oury

LRI, Univ. Paris-Sud - Démons Team & INRIA Saclay - ProVal Project

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Solutions for overloading

- Intersection types: overloading by declaring multiple signatures for a single constant (e.g. $\mathbb{C}$Duce).
- Bounded quantification and class-based overloading. Overloading circumscribed by a subtyping relation (e.g. structural subtyping à la OCaml).

Solutions for overloading

- Intersection types: overloading by declaring multiple signatures for a single constant (e.g. $\mathbb{C}$Duce).
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Our objective:

- Modularity: separate definitions of the specializations
- The setting is Coq: no intentional type analysis, no latitude on the kernel language!

Making ad-hoc polymorphism less *ad hoc*

```
class Eq a where
  (==) :: a → a → Bool

instance Eq Bool where
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in :: Eq a ⇒ a → [a] → Bool
in x [] = False
in x (y : ys) = x == y || in x ys
```

Parameterized instances

```
instance (Eq a) ⇒ Eq [a] where
  [] == []                 = True
  (x : xs) == (y : ys)     = x == y && xs == ys
  _ == _                   = False
```

A structuring concept

```
class Num a where
  (+) :: a → a → a ...
class (Num a) ⇒ Fractional a where
  (/) :: a → a → a ...
class (Fractional a) ⇒ Floating a where
  exp :: a → a ...
```

The MLer point of view

A system of modules and functors with sugar for implicit instantiation and functorization.

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```

- Extensions: dependent types give new power to type classes.

```
Class Reflexive A (R : relation A) :=
  reflexive : ∀ x, R x x.
```

Outline

1. Type Classes in Coq
   - A cheap implementation
   - Example: Numbers and monads
2. Superclasses and substructures
   - The power of Pi
   - Example: Categories
3. Extensions
   - Dependent classes
   - Logic Programming
4. Summary, Related, Current and Future Work

Ingredients

- Dependent records: a singleton inductive type containing each component and some projections.

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- Implicit arguments: inferring the value of arguments (e.g. types).

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Check (@id : Π A, A → A).
Check (@id nat : nat → nat).
```

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Check (id : nat → nat).

Check (id 3).
```

Implementation

- Parameterized dependent records

  $\texttt{Class}\ \mathsf{Id}\ (\alpha_1 : \tau_1) \cdots (\alpha_n : \tau_n) :=$
  $\quad \mathrm{f}_1 : \phi_1\ ;\ \cdots\ ;\ \mathrm{f}_m : \phi_m.$

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$$\texttt{Record}\ \mathsf{Id}\ (\alpha_1 : \tau_1) \cdots (\alpha_n : \tau_n) :=$$
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  $\mathrm{f}_1 : \forall \overrightarrow{\alpha_n : \tau_n}, \mathsf{Id}\ \overrightarrow{\alpha_n} \to \phi_1$

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  $\mathrm{f}_1 : \forall \{\overrightarrow{\alpha_n : \tau_n}\}, \{\mathsf{Id}\ \overrightarrow{\alpha_n}\} \to \phi_1$

- Proof-search tactic with instances as lemmas

  $A : \texttt{Type}, \mathit{eqa} : \mathsf{Eq}\ A \vdash ? : \mathsf{Eq}\ (\mathsf{list}\ A)$

Elaboration with classes, an example

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Elaboration with classes, an example

$$
\begin{aligned}
&(\lambda x\ y : \mathsf{bool}.\ \mathrm{eqb}\ x\ y) \\
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\rightsquigarrow\ &\{\ \text{proof search for } \mathsf{Eq}\ \mathsf{bool} \text{ returns } \mathrm{Eq\_bool}\ \} \\
&(\lambda x\ y : \mathsf{bool}.\ @\mathrm{eqb}\ \mathsf{bool}\ \mathrm{Eq\_bool}\ x\ y)
\end{aligned}
$$

Outline

Numeric overloading

$$\texttt{Class}\ \mathsf{Num}\ \alpha := \mathrm{zero} : \alpha\ ;\ \mathrm{one} : \alpha\ ;\ \mathrm{plus} : \alpha \to \alpha \to \alpha.$$

Numeric overloading

```
Class Num α := zero : α ; one : α ; plus : α → α → α.

Instance nat_num : Num nat :=
  zero := 0%nat ; one := 1%nat ; plus := Peano.plus.

Instance Z_num : Num Z :=
  zero := 0%Z ; one := 1%Z ; plus := Zplus.
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Notation "0" := zero.
Notation "1" := one.
Infix "+" := plus.
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Notation "0" := zero.
Notation "1" := one.
Infix "+" := plus.

Check (λ x : nat, x + (1 + 0 + x)).
Check (λ x : Z, x + (1 + 0 + x)).
```

Monad

```
Class Monad (η : Type → Type) :=
  unit : ∀ {α}, α → η α ;
  bind : ∀ {α β}, η α → (α → η β) → η β ;

  bind_unit_left : ∀ α β (x : α) (f : α → η β),
    bind (unit x) f = f x ;
  bind_unit_right : ∀ α (x : η α), bind x unit = x ;
  bind_assoc : ∀ α β δ
    (x : η α) (f : α → η β) (g : β → η δ),
    bind x (fun a : α ⇒ bind (f a) g) = bind (bind x f) g.
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    (x : η α) (f : α → η β) (g : β → η δ),
    bind x (fun a : α ⇒ bind (f a) g) = bind (bind x f) g.

Infix "≫=" := bind (at level 55).
Notation "x ← T ; E" := (bind T (fun x : _ ⇒ E))
  (at level 30, right associativity).
Notation "'return' t" := (unit t) (at level 20).
```

Definitions

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Program Instance identity_monad : Monad id :=
  unit α a := a ;
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Section Monad_Defs.
  Context [ mon : Monad η ].
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Definitions

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Section Monad_Defs.
  Context [ mon : Monad η ].

  Definition ap {α β} (f : α → β) (x: η α) : η β :=
    a ← x ; return (f a).

  Definition join {α} (x : η (η α)) : η α :=
    x ≫= id.
```

Proofs

```
Lemma do_return_eta : ∀ α (u : η α),
    x ← u ; return x = u.
Proof. intros α u. rewrite ← (eta_expansion unit).
```

```
η : Type → Type
mon : Monad η
α : Type
u : η α
============================
 u >>= unit = u
```

Proofs

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```
η : Type → Type
mon : Monad η
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============================
 u >>= unit = u
```

```
  apply bind_unit_right.
Qed.

End Monad_Defs.
```

Outline

Fields or Parameters ?

When one doesn't have manifest types and `with` constraints...

```
Class Functor :=
  A : Type; B : Type;
  src : Category A ; dst : Category B ; ...

Class Functor obj obj' :=
  src : Category obj ; dst : Category obj' ; ...

Class Functor obj (src : Category obj) obj' (dst : Category obj')

  := ...
```

???

Sharing by equalities

```
Definition adjunction (F : Functor) (G : Functor),
  src F = dst G → dst F = src G ...
```

Obfuscates the goals and the computations, awkward to use.

Sharing by parameters

$$\texttt{Class}\ \{(C : \text{Category}\ obj,\ D : \text{Category}\ obj')\} \Rightarrow \text{Functor} := \ldots$$

$$\equiv$$

$$\texttt{Class}\ \text{Functor}\ \{(C : \text{Category}\ obj,\ D : \text{Category}\ obj')\} := \ldots$$

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$$\texttt{Class}\ \text{Functor}\ \{(C : \text{Category}\ obj, D : \text{Category}\ obj')\} := \ldots$$

$$\equiv$$

$$\texttt{Record}\ \text{Functor}\ \{obj\}\ (C : \text{Category}\ obj)$$
$$\quad \{obj'\}(D : \text{Category}\ obj') := \ldots$$

Sharing by parameters

```
Class {(C : Category obj, D : Category obj')} ⇒ Functor := ...
```

≡

```
Class Functor {(C : Category obj, D : Category obj')} := ...
```

≡

```
Record Functor {obj} (C : Category obj)
    {obj'}(D : Category obj') := ...

Definition adjunction [ C : Category obj, D : Category obj' ]
    (F : Functor C D) (G : Functor D C) := ...
```

Uses the dependent product and named, first-class instances.

Implicit Generalization

An old convention: the free variables of a statement are implicitly universally quantified. E.g., when defining a set of equations:

$$\begin{aligned} x + y &= y + x \\ x + 0 &= 0 \\ x + \mathsf{S}\, y &= \mathsf{S}\,(x + y) \end{aligned}$$

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$$
\begin{array}{lcl}
x + y & = & y + x \\
x + 0 & = & 0 \\
x + \mathsf{S}\ y & = & \mathsf{S}\ (x + y)
\end{array}
$$

We introduce new syntax to automatically generalize the free variables of a given term or binder:

$$
\begin{array}{lcl}
\Gamma \vdash \text{`}(t) : \texttt{Type} & \triangleq & \Gamma \vdash \Pi_{\mathcal{FV}(t)\setminus\Gamma}, t \\
\Gamma \vdash \text{`}(t) : T : \texttt{Type} & \triangleq & \Gamma \vdash \lambda_{\mathcal{FV}(t)\setminus\Gamma}, t \\
\overrightarrow{(x_i : \tau_i)}\ \{(y : T)\} & \triangleq & \overrightarrow{(x_i : \tau_i)}\ \{(\mathcal{FV}(T) \setminus \overrightarrow{x_i})\}\ (y : T) \\
\overrightarrow{(x_i : \tau_i)}\ ((y : T)) & \triangleq & \overrightarrow{(x_i : \tau_i)}\ (\mathcal{FV}(T) \setminus \overrightarrow{x_i})\ (y : T)
\end{array}
$$

Substructures

A **superclass** becomes a parameter, a **substructure** is a method which is also an instance.

```
Class Monoid A :=
  monop : A → A → A ; ...

Class Group A :=
  grp_mon :> Monoid A ; ...
```

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Instance grp_mon [ Group A ] : Monoid A.
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Class Group A :=
  grp_mon : Monoid A ; ...

Instance grp_mon [ Group A ] : Monoid A.

Definition foo [ Group A ] (x : A) : A := monop x x.
```

Similar to the existing `Structure`s based on coercive subtyping.

Outline

Category

```
Class Category (obj : Type) (hom : obj → obj → Type) :=
  morphisms :> ∀ a b, Setoid (hom a b) ;
  id : ∀ a, hom a a;
  compose : ∀ {a b c}, hom a b → hom b c → hom a c;
  id_unit_left : ∀ ((f : hom a b)), compose f (id b) == f;
  id_unit_right : ∀ ((f : hom a b)), compose (id a) f == f;
  assoc : ∀ a b c d (f : hom a b) (g : hom b c) (h : hom c d),
    compose f (compose g h) == compose (compose f g) h.

Notation " x '∘' y " := (compose y x)
  (left associativity, at level 40).
```

Abstract instances

```
Definition opposite (X : Type) := X.

Program Instance opposite_category {( Category obj hom )} :
  Category (opposite obj) (flip hom).
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Class {(C : Category obj hom)} ⇒ Terminal (one : obj) :=
  bang : ∀ x, hom x one ;
  unique : ∀ x (f g : hom x one), f == g.
```

An abstract proof

```
Definition isomorphic [ Category obj hom ] a b :=
  { f : hom a b & { g : hom b a |
    f ∘ g == id b ∧ g ∘ f == id a } }.

Lemma terminal_isomorphic [ C : Category obj hom ] :
  ‘( Terminal C x → Terminal C y → isomorphic x y ).
Proof.
  intros. red.
  do 2 ∃ (bang _).
  split ; apply unique.
Qed.
```

Outline

Dependent classes (demo script)

```
Class Reflexive {A} (R : relation A) := refl : Π x, R x x.

Print Reflexive.

Instance eq_refl A : Reflexive (@eq A).
Proof. red. apply refl_equal. Qed.

Instance iff_refl : Reflexive iff.
Proof. red. tauto. Qed.

Goal Π P, P ↔ P.
Proof. apply refl. Qed.

Goal Π A (x : A), x = x.
Proof. intros A ; apply refl. Qed.

Ltac reflexivity' := apply refl.

Lemma foo [ Reflexive nat R ] : R 0 0.
Proof. intros. reflexivity'. Qed.
```

Boolean formulas

```
Inductive formula :=
| cst : bool → formula
| not : formula → formula
| and : formula → formula → formula
| or : formula → formula → formula
| impl : formula → formula → formula.
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Fixpoint interp f :=
  match f with
    | cst b ⇒ if b then True else False
    | not b ⇒ ¬ interp b
    | and a b ⇒ interp a ∧ interp b
    | or a b ⇒ interp a ∨ interp b
    | impl a b ⇒ interp a → interp b
  end.
```

Reification

```
Class Reify (prop : Prop) :=
  reification : formula ;
  reify_correct : interp reification ↔ prop.
```

Reification

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Check (@reification : Π prop : Prop, Reify prop → formula).

Implicit Arguments reification [[Reify]].
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Implicit Arguments reification [[Reify]].

Program Instance true_reif : Reify True :=
  reification := cst true.
Program Instance not_reif [ Rb : Reify b ] : Reify (¬ b) :=
  reification := not (reification b).
```

Reification

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Program Instance not_reif [ Rb : Reify b ] : Reify (¬ b) :=
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Example example_prop :=
  reification ((True ∧ ¬ False) → ¬ ¬ False).

Check (refl_equal _ : example_prop =
  impl (and (cst true) (not (cst false))) (not (not (cst false)))).
```

Outline

Summary

- ✓ A **lightweight** and **general** implementation of type classes, "available" in Coq v8.2
- ✓ A type-theoretic **explanation** and **extension** of type-classes concepts

On top of that:

- Realistic test-case: a new setoid-rewriting tactic built on top of classes.
- A system to automatically infer instances by Matthias Puech.

How does it compare to Canonical Structures?

- Declaration of a Class and instances instead of using implicit coercions + declaration of some canonical structures.

```
Class Coercion (from to : Type) :=
  coerce : from → to.
```

How does it compare to Canonical Structures?

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- Indexing on parameters only but less sensible to the shape of unification problems (simpler to explain!).

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- Declaration of a Class and instances instead of using implicit coercions + declaration of some canonical structures.
- Indexing on parameters only but less sensible to the shape of unification problems (simpler to explain!).
- Based on an extensible resolution system instead of recursive unification of head constants.

Ongoing and future work

- Debugging!
- Refined, parameterized proof-search (ambiguity checking, mode declarations, discrimination nets . . .)
- Integration with the proof shell: move to **open** terms
- Improve extraction and embedding of HASKELL programs

The End

http://coq.inria.fr/V8.2beta/