PROGRAM-ing in COQ

MATTHIEU SOZEAU
under the direction of CHRISTINE PAULIN-MOHRING

LRI, Univ. Paris-Sud - DÉMONS Team & INRIA SACLAY - PROVAL Project

Harvard PL Seminar
January 23th 2008
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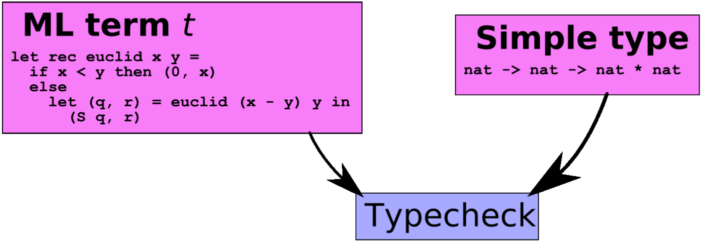
The Big Picture

ML term *t*

```
let rec euclid x y =
  if x < y then (0, x)
  else
    let (q, r) = euclid (x - y) y in
      (S q, r)
```

The Big Picture



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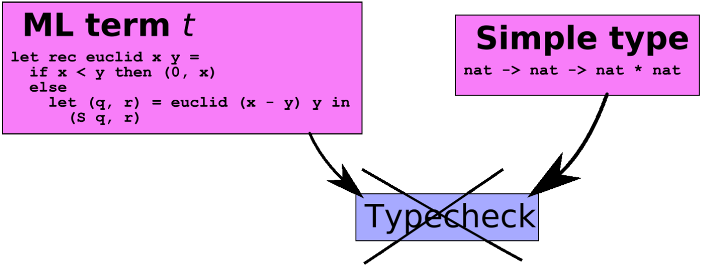
Simple type

```
nat -> nat -> nat * nat
```

Typecheck

The Big Picture



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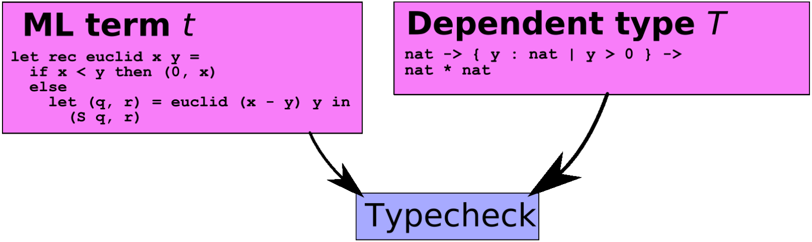

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Dependent type T

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nat -> { y : nat | y > 0 } ->
nat * nat
```

Typecheck

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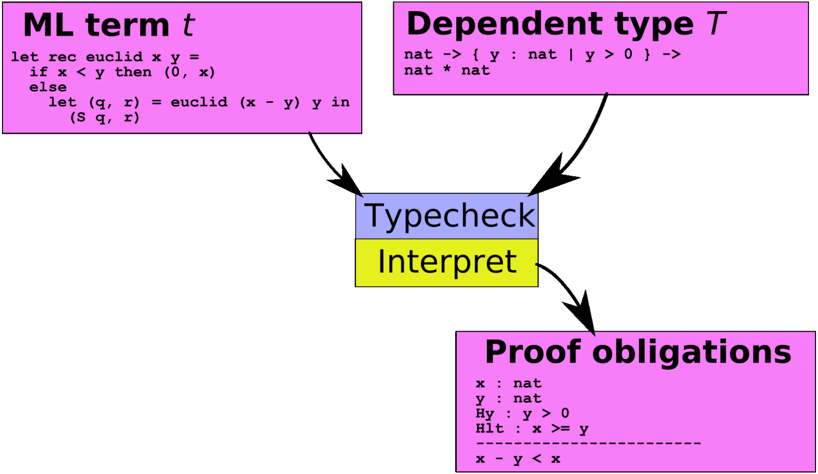

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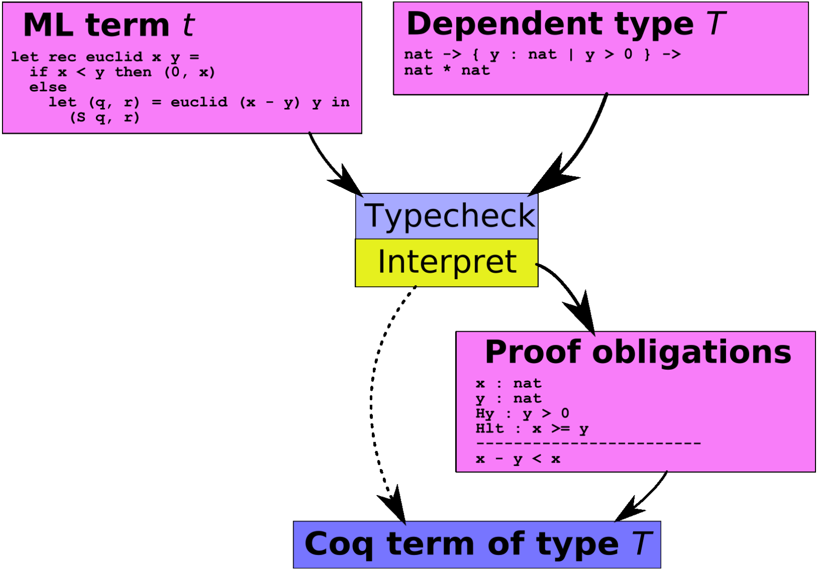
Typecheck

Interpret

Proof obligations

```
x : nat
y : nat
Hy : y > 0
Hlt : x >= y
------------------------
x - y < x
```

The Big Picture



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Coq term of type *T*

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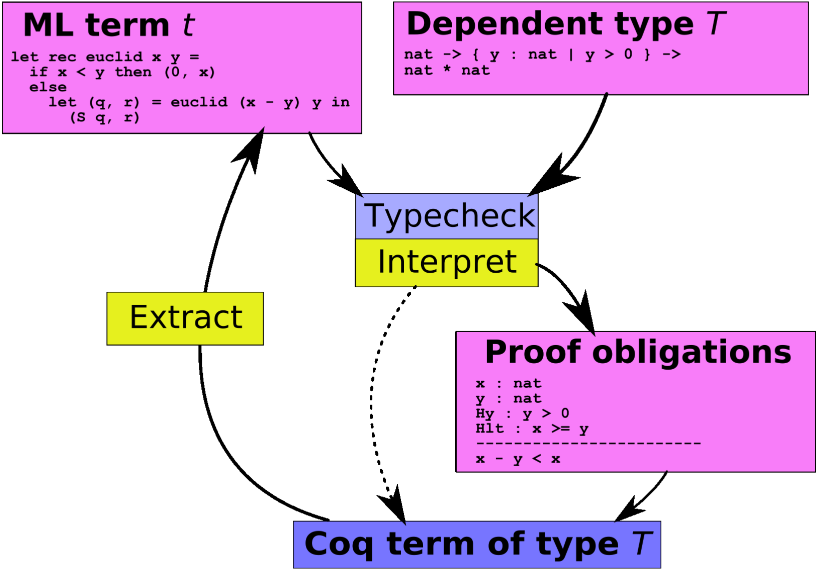

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Interpret

Extract

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Hy : y > 0
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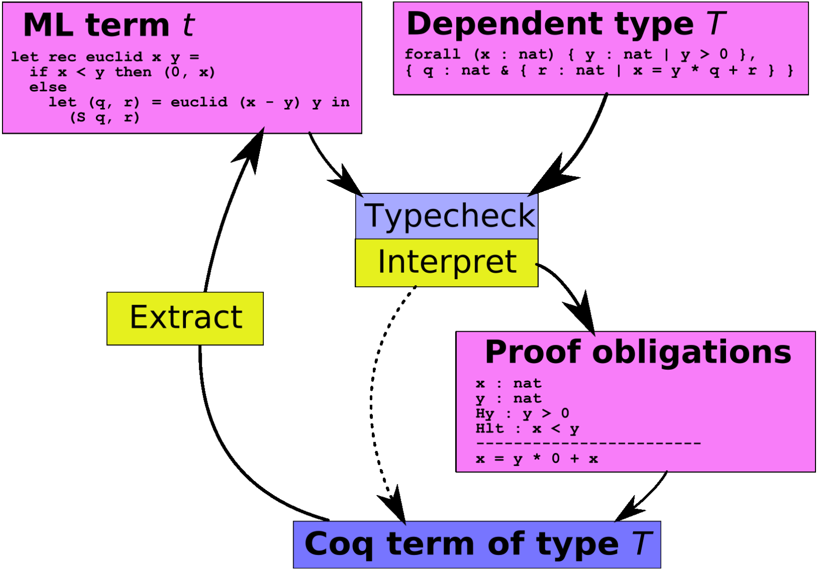
Coq term of type *T*

The Big Picture

```
Inductive diveucl a b : Set :=
   divex : ∀ q r, b > r → a = q × b + r → diveucl a b.

Lemma eucl_dev : ∀ n, n > 0 → ∀ m:nat, diveucl m n.
Proof.
   intros b H a; pattern a in ⊢ ×; apply gt_wf_rec; intros n H0.
   elim (le_gt_dec b n).
   intro lebn.
   elim (H0 (n - b)); auto with arith.
   intros q r g e.
   apply divex with (S q) r; simpl in ⊢ ×; auto with arith.
   elim plus_assoc.
   elim e; auto with arith.
   intros gtbn.
   apply divex with 0 n; simpl in ⊢ ×; auto with arith.
Qed.
```

The Big Picture



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Dependent type *T*

```
forall (x : nat) { y : nat | y > 0 },
{ q : nat & { r : nat | x = y * q + r } }
```

Typecheck

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Proof obligations

```
x : nat
y : nat
Hy : y > 0
Hlt : x < y
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x = y * 0 + x
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Coq term of type *T*

The Curry-Howard isomorphism

Programming language = Proof system

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Program extends the Coq proof-assistant into a dependently-typed programming environment.

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Epigram **PVS** **DML** **Ωmega**

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Programming language = Proof system

PROGRAM extends the COQ proof-assistant into a dependently-typed programming environment.

- **Logical Framework** Type Theory.

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- **Development style and proof automation** Interactive, semi-automatic proof using tactics.

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- **Phase distinction** none

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- **Development style and proof automation** Interactive, semi-automatic proof using tactics.
- **Phase distinction** ⇒ in PROGRAM

~~**Epigram**~~ $\boxed{\textbf{PVS}}$ ~~**DML**~~ ~~**Ωmega**~~

A simple idea

Definition

$\{x : T \mid P\}$ is the set of objects of set T verifying property P.

- Useful for specifying, widely used in mathematics ;
- Links object and property.

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Adapting the idea

$$\frac{t : T \qquad P[t/x]}{t : \{\ x : T \mid P\ \}} \qquad \frac{t : \{\ x : T \mid P\ \}}{t : T}$$

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Adapting the idea

$$\frac{t : T \qquad p : P[t/x]}{(t, p) : \{\ x : T \mid P\ \}} \qquad \frac{t : \{\ x : T \mid P\ \}}{\texttt{proj}\ t : T}$$

From "*Predicate subtyping*" . . .

PVS

- Specialized typing algorithm for subset types, generating *Type-checking conditions*.

$t : \{\ x : T \mid P\ \}$	used as	$t : T$	ok
$t : T$	used as	$t : \{\ x : T \mid P\ \}$	if $P[t/x]$

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+ Practical success ;

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+ Practical success ;

– No strong safety guarantee in PVS.

. . . to Subset coercions

1. A property-irrelevant language (Russell) with decidable typing ;

$$\frac{\Gamma \vdash t : \{\, x : T \mid P \,\}}{\Gamma \vdash \qquad t : T}$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma, x : T \vdash P : \texttt{Prop}}{\Gamma \vdash \; t \quad : \{\, x : T \mid P \,\}}$$

. . . to Subset coercions

1. A property-irrelevant language (RUSSELL) with decidable typing ;
2. A total interpretation to COQ terms with holes ;

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. . . to Subset coercions

1. A property-irrelevant language (RUSSELL) with decidable typing ;
2. A total interpretation to COQ terms with holes ;
3. A mechanism to turn the holes into proof obligations and manage them.

$$\frac{\Gamma \vdash t : \{\ x : T \mid P\ \}}{\Gamma \vdash \texttt{proj}\ t : T}$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma, x : T \vdash P : \texttt{Prop} \qquad \Gamma \vdash p : P[t/x]}{\Gamma \vdash (t, p) : \{\ x : T \mid P\ \}}$$

RUSSELL syntax

$$
\begin{array}{lcl}
x & \in & \mathcal{V} \\
\\
s, t, u, v & ::= & x \\
 & \mid & \texttt{Set} \\
 & \mid & \texttt{Prop} \\
 & \mid & \texttt{Type}
\end{array}
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RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \equiv_{\beta\pi} T : s}{\Gamma \vdash t : T}$$

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Example $$\frac{\Gamma \vdash 0 : \mathbb{N} \qquad \Gamma \vdash \mathbb{N} \rhd \{\, x : \mathbb{N} \mid x \neq 0 \,\} : \texttt{Set}}{\Gamma \vdash 0 : \{\, x : \mathbb{N} \mid x \neq 0 \,\}}$$

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$\textsc{Russell}$ typing $\vdash$ and coercion $\rhd$

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$$\frac{\Gamma \vdash U \rhd T : s_1 \qquad \Gamma, x : U \vdash V \rhd W : s_2}{\Gamma \vdash \Pi x : T.V \rhd \Pi x : U.W : s_2}$$

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$\rhd$ is symmetric!

Results

Theorem (Decidability of type checking and type inference)

$\Gamma \vdash t : T$ *is decidable.*

$$\frac{\Gamma \vdash f : T \quad \Gamma \vdash T \rhd \Pi x : A.B : s \qquad \Gamma \vdash e : E \qquad \Gamma \vdash E \rhd A : s'}{\Gamma \vdash (f\ e) : B[e/x]}$$

Results

Theorem (Decidability of type checking and type inference)

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COQ corner

Mechanised proofs of Subject Reduction and equivalence between declarative and algorithmic presentations of the system.

From RUSSELL to COQ

The target system : CIC with metavariables

$$\frac{\Gamma \vdash_{?} t : T \qquad \Gamma \vdash_{?} p : P[t/x]}{\Gamma \vdash_{?} \mathbf{elt}\ T\ P\ t\ p : \{\ x : T \mid P\ \}}$$

$$\frac{\Gamma \vdash_{?} t : \{\ x : T \mid P\ \}}{\Gamma \vdash_{?} \sigma_1\ t : T} \qquad \frac{\Gamma \vdash_{?} t : \{\ x : T \mid P\ \}}{\Gamma \vdash_{?} \sigma_2\ t : P[\sigma_1\ t/x]}$$

$$\frac{\Gamma \vdash_{?} P : \texttt{Prop}}{\Gamma \vdash_{?} ?_P : P}$$

We build an interpretation $[\![\_]\!]_\Gamma$ from RUSSELL to CIC$_?$ terms.

From RUSSELL to COQ

The target system : CIC with metavariables

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We build an interpretation $[\![\_]\!]_\Gamma$ from RUSSELL to $\text{CIC}_?$ terms.

Our goal

If $\Gamma \vdash t : T$ then $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$.

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies $[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma$.

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Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \quad : T \rhd U}$$

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$$\Gamma \vdash_? \qquad : \{\, x : T \mid P \,\} \rhd T$$

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$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \sigma_1\ \bullet : \{\ x : T \mid P\ \} \rhd T$$

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$$\Gamma \vdash_? \qquad\qquad : T \rhd \{ \; x : T \mid P \; \}$$

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$$\Gamma \vdash_? \mathbf{elt}\ \_\ \_\ \bullet\ ?_{[\![P]\!]_{\Gamma, x:T}[\bullet/x]} : T \rhd \{ x : T \mid P \}$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies $[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma$.

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \sigma_1\ \bullet : \{\ x : T \mid P\ \} \rhd T$$

$$\Gamma \vdash_? \textbf{elt}\ \_\ \_\ \bullet\ ?_{[\![P]\!]_{\Gamma,x:T}[\bullet/x]} : T \rhd \{\ x : T \mid P\ \}$$

Example

$$\frac{\Gamma \vdash_? 0 : \mathbb{N} \qquad \Gamma \vdash_? \textbf{elt}\ \_\ \_\ \bullet\ ?_{(x\neq 0)[\bullet/x]} : \mathbb{N} \rhd \{\ x : \mathbb{N} \mid x \neq 0\ \}}{\Gamma \vdash_? \textbf{elt}\ \_\ \_\ 0\ ?_{0\neq 0} : \{\ x : \mathbb{N} \mid x \neq 0\ \}}$$

Interpretation of terms

Example (Application)

$$\frac{\Gamma \vdash f : T \quad \Gamma \vdash T \rhd \Pi x : V.W : s \qquad \Gamma \vdash u : U \qquad \Gamma \vdash U \rhd V : s'}{\Gamma \vdash (f\ u) : W[u/x]}$$

$$\begin{array}{rcl} [\![f\ u]\!]_\Gamma & \triangleq & \textbf{let}\ \pi = \textbf{coerce}_\Gamma\ T\ (\Pi x : V.W)\ \textbf{in} \\ & & \textbf{let}\ c = \textbf{coerce}_\Gamma\ U\ V\ \textbf{in} \\ & & (\pi[[\![f]\!]_\Gamma])\ (c[[\![u]\!]_\Gamma]) \end{array}$$

Theorem (Soundness)

If $\Gamma \vdash t : T$ *then* $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$.

Theoretical matters . . .

$\vdash_?$'s equational theory:

$$\begin{array}{llcll}
(\beta) & (\lambda x : X.e)\ v & \equiv & e[v/x] & \\
(\pi_i) & \pi_i\ (e_1, e_2)_T & \equiv & e_i & \\
(\sigma_i) & \sigma_i\ (\mathbf{elt}\ E\ P\ e_1\ e_2) & \equiv & e_i & \\
(\eta) & (\lambda x : X.e\ x) & \equiv & e & \text{if } x \notin FV(e) \\
(\mathrm{SP}) & \mathbf{elt}\ E\ P\ (\sigma_1\ e)\ (\sigma_2\ e) & \equiv & e &
\end{array}$$

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(\mathrm{SP}) & \mathbf{elt}\ E\ P\ (\sigma_1\ e)\ (\sigma_2\ e) & \equiv & e & \\
(PI) & \mathbf{elt}\ E\ P\ t\ p & \equiv & \mathbf{elt}\ E\ P\ t'\ p' & \text{if } t \equiv t'
\end{array}$$

⇒ Proof Irrelevance

Theoretical matters . . .

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\end{array}$$

⇒ Proof Irrelevance

. . . have practical effects

Difficulty to reason on code: $\mathbf{elt}\ T\ P\ x\ p_1 \not\equiv \mathbf{elt}\ T\ P\ x\ p_2$ where $p_1, p_2 : P\ x$.

Inductive types

Different representations

vector $n \triangleq \{\ x :$ list $A \mid$ *length* $x = n\ \}$ or
vector $n \triangleq$ vnil : vector $0 \mid$ vcons : $A \rightarrow \forall n,$ vector $n \rightarrow$ vector $($S $n)$?

Inductive types

Different representations

vector $n \triangleq \{\ x : \textsf{list}\ A \mid \textit{length}\ x = n\ \}$ or
vector $n \triangleq \textsf{vnil} : \textsf{vector}\ 0 \mid \textsf{vcons} : A \to \forall n, \textsf{vector}\ n \to \textsf{vector}\ (\textsf{S}\ n)$?

$$\frac{\Gamma \vdash v : \textsf{vector}\ x \qquad \dfrac{\Gamma \vdash x = y : \texttt{Prop}}{\Gamma \vdash \textsf{vector}\ x \rhd \textsf{vector}\ y : \texttt{Set}}}{\Gamma \vdash v : \textsf{vector}\ y}$$

The PROGRAM vernacular

Architecture

Wrap around COQ's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, . . .).

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Wrap around COQ's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, …).

1. Use the COQ parser.

`Program Definition` $f \ : \ T \quad := \ t$.

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the Coq parser.
2. Typecheck $\Gamma \vdash t : T$ and generate $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$;

$$\texttt{Program Definition}\ f\ : [\![T]\!]_\Gamma := [\![t]\!]_\Gamma \qquad .$$

The PROGRAM vernacular

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Wrap around COQ's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the COQ parser.
2. Typecheck $\Gamma \vdash t : T$ and generate $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$;
3. Interactive proving of obligations ;

`Program Definition` $f \ : [\![T]\!]_\Gamma := [\![t]\!]_\Gamma +$ obligations.

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Wrap around COQ's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, . . .).

1. Use the COQ parser.
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3. Interactive proving of obligations ;
4. Final definition.

`Definition` $f\ : [\![T]\!]_\Gamma := [\![t]\!]_\Gamma +$ obligations.

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Wrap around COQ's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, …).

1. Use the COQ parser.
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4. Final definition.

Restriction We assume $\Gamma \vdash_{CCI} [\![T]\!]_\Gamma : s$.

`Definition` $f \; : [\![T]\!]_\Gamma := [\![t]\!]_\Gamma +$ obligations.

Hello world: Euclidean division

DEMO

Pattern-matching revisited

Put logic into the terms.

Let $e : \mathbb{N}$:

$$\begin{array}{lll} \textbf{match } e & \textbf{return} & T \textbf{ with} \\ \mid \mathsf{S}\ n \Rightarrow & & t_1 \\ \mid 0 \Rightarrow & & t_2 \\ \textbf{end} & & \end{array}$$

Pattern-matching revisited

Put logic into the terms.

Let $e : \mathbb{N}$:

$$\begin{array}{ll}
\textbf{match } e \textbf{ as } t & \textbf{return } t = e \to T \textbf{ with} \\
\mid \mathsf{S}\ n \Rightarrow & \textbf{fun } (H : \mathsf{S}\ n = e) \Rightarrow t_1 \\
\mid 0 \Rightarrow & \textbf{fun } (H : 0 = e) \Rightarrow t_2 \\
\textbf{end} & (\textbf{refl\_equal } e)
\end{array}$$

Pattern-matching revisited

Put logic into the terms.

Further refinements

- Each branch typed only once ;

Let $e : \mathbb{N}$:

$$\begin{array}{ll}
\textbf{match } e \textbf{ as } t & \textbf{return } t = e \to T \textbf{ with} \\
\mid \mathsf{S}\ (\mathsf{S}\ n) \Rightarrow & \textbf{fun } (H : \mathsf{S}\ (\mathsf{S}\ n) = e) \Rightarrow t_1 \\
\mid n \Rightarrow & \textbf{fun } (H : n = e) \Rightarrow t_2 \\
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\mid \mathsf{S}\ 0 \Rightarrow & \textbf{fun } (H : \mathsf{S}\ 0 = e) \Rightarrow\ t_2 \\
\mid 0 \Rightarrow & \textbf{fun } (H : 0 = e) \Rightarrow t_2 \\
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- Each branch typed only once ;
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\mid \mathsf{S}\ (\mathsf{S}\ n) \Rightarrow & \textbf{fun } (H : \mathsf{S}\ (\mathsf{S}\ n) = e) \Rightarrow t_1 \\
\mid n \Rightarrow & \textbf{fun } (H : n = e) \Rightarrow \textbf{let } H' : \forall n', n \neq \mathsf{S}\ (\mathsf{S}\ n') \textbf{ in } t_2 \\
\textbf{end} & (\textbf{refl\_equal } e)
\end{array}$$

Pattern-matching revisited

Put logic into the terms.

Further refinements

- Each branch typed only once ;
- Add inequalities for intersecting patterns ;
- Generalized to dependent inductive types.

Let e : vector n:

match e **return** T **with**
| vnil ⇒ t_1
| vcons x n' v' ⇒ t_2
end

Pattern-matching revisited

Put logic into the terms.

Further refinements

- Each branch typed only once ;
- Add inequalities for intersecting patterns ;
- Generalized to dependent inductive types.

Let e : vector n:

$$
\begin{array}{l}
\textbf{match } e \textbf{ as } t \textbf{ in } \text{vector } n' \textbf{ return } n' = n \rightarrow t \simeq e \rightarrow T \textbf{ with} \\
\mid \text{vnil} \Rightarrow \textbf{fun } (H : 0 = n)(Hv : \text{vnil} \simeq e) \Rightarrow t_1 \\
\mid \text{vcons } x\ n'\ v' \Rightarrow \textbf{fun } (H : \mathsf{S}\ n' = n)(Hv : \text{vcons } x\ n'\ v' \simeq e) \Rightarrow t_2 \\
\textbf{end}(\text{refl\_equal } n)(\text{JMeq\_refl } e)
\end{array}
$$

Sugar

Obligations

Unresolved implicits (_) are turned into obligations, à la `refine`.

Bang

$! \triangleq (\texttt{False\_rect}\ \_\ \_)$ where $\texttt{False\_rect} : \forall A : \texttt{Type}, \texttt{False} \to A$. It corresponds to ML's `assert(false)`.

$$\textbf{match}\ 0\ \textbf{with}\ 0 \Rightarrow 0 \mid n \Rightarrow\ !\ \textbf{end}$$

Sugar

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Unresolved implicits (_) are turned into obligations, à la `refine`.

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$! \triangleq (\texttt{False\_rect}\ \_\ \_)$ where $\texttt{False\_rect} : \forall A : \texttt{Type}, \texttt{False} \rightarrow A$. It corresponds to ML's `assert(false)`.

$$\textbf{match}\ 0\ \textbf{with}\ 0 \Rightarrow 0 \mid n \Rightarrow !\ \textbf{end}$$

Destruction

Let $\textbf{dest}\ t\ \textbf{as}\ p\ \textbf{in}\ e \triangleq \textbf{match}\ t\ \textbf{with}\ p \Rightarrow e\ \textbf{end}$. p can be an arbitrary pattern.

Recursion

Support for well-founded recursion and measures.

$$\texttt{Program Fixpoint}\ f\ (a : \mathbb{N})\ \{\textbf{wf} < a\} : \mathbb{N} := b.$$

Recursion

Support for well-founded recursion and measures.

$$\texttt{Program Fixpoint}\ f\ (a : \mathbb{N})\ \{\textbf{wf} < a\} : \mathbb{N} := b.$$

$$\frac{\begin{array}{lcl} a & : & \mathbb{N} \\ f & : & \{x : \mathbb{N} \mid x < a\} \to \mathbb{N} \end{array}}{\begin{array}{lcl} b & : & \mathbb{N} \end{array}}$$

Safe Lists

DEMO

A quick tour of Finger Trees

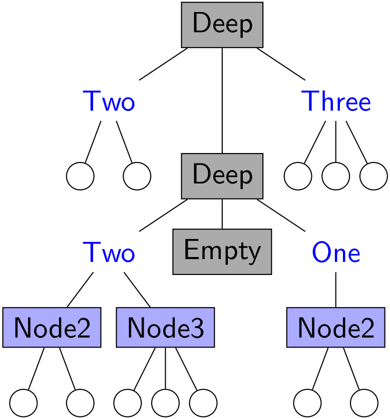
- A Simple General Purpose Data Structure (Hinze & Paterson, JFP 2006)
- Purely functional, nested datatype
- Parameterized data structure
- Efficient deque operations, concatenation and splitting
- Comparable to Kaplan & Tarjan's catenable deques

The Big Finger Tree Picture

`data` Digit a = One a | Two a a | Three a a a | Four a a a a

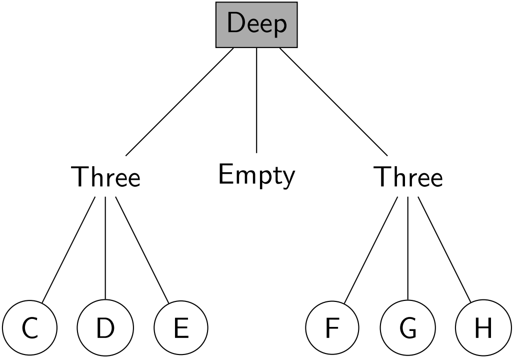
`data` Node a = Node2 a a | Node3 a a a

`data` FingerTree a =
  | Empty
  | Single a
  | Deep
    (Digit a)
    (FingerTree (Node a))
    (Digit a)



Operating on a Finger Tree

$add\_left :: a \rightarrow \mathrm{FingerTree}\ a \rightarrow \mathrm{FingerTree}\ a$
$add\_left\ a\ \mathrm{Empty} = \mathrm{Single}\ a$
$add\_left\ a\ (\mathrm{Single}\ b) = \mathrm{Deep}\ (\mathrm{One}\ a)\ \mathrm{Empty}\ (\mathrm{One}\ b)$
$add\_left\ a\ (\mathrm{Deep}\ pr\ m\ sf) = \ldots$



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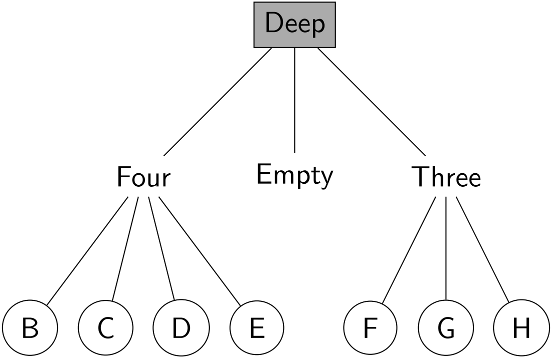
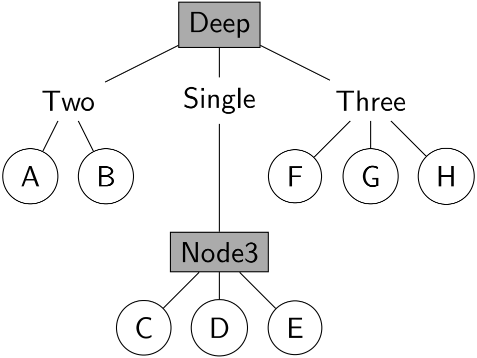


Operating on a Finger Tree

```
add_left :: a → FingerTree a → FingerTree a
add_left a Empty = Single a
add_left a (Single b) = Deep (One a) Empty (One b)
add_left a (Deep pr m sf) = ...
```



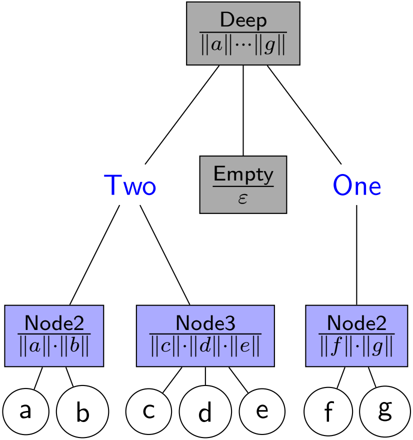
Adding cached measures

```
class Monoid v ⇒ Measured v a where
   ‖_‖ :: a → v

instance (Measured v a) ⇒ Measured v (Digit a) where ···
```

```
data Node v a =
   Node2 v a a | Node3 v a a a

data FingerTree v a =
   | Empty
   | Single a
   | Deep v
      (Digit a)
      (FingerTree v (Node v a))
      (Digit a)
```



Outline

1. The idea
    - A simple idea
    - From PVS to COQ
2. Theoretical development
    - RUSSELL
    - Interpretation in COQ
    - Inductive types
3. PROGRAM
    - Architecture
    - Hello world
    - Extensions
4. Finger Trees
    - In HASKELL
    - In COQ
5. Conclusion

Why do this ?

- Generally useful, non-trivial structure
- Abstraction power needed to ensure coherence of measures
- Makes dependent types (subsets and indexed datatypes) shine
- Fun ! Helps solve the ICFP contest using COQ

Digits

```
Variable A : Type.

Inductive digit : Type :=
| One : A → digit
| Two : A → A → digit
| Three : A → A → A → digit
| Four : A → A → A → A → digit.

Definition full x :=
  match x with Four _ _ _ _ ⇒ True | _ ⇒ False end.
```

Digits cont'd

```
Program Definition add_digit_left
  (a : A) (d : digit | ¬ full d) : digit :=
  match d with
    | One x ⇒ Two a x
    | Two x y ⇒ Three a x y
    | Three x y z ⇒ Four a x y z
    | Four _ _ _ _ ⇒ !
  end.
Next Obligation.
  intros ; simpl in n ; auto.
Qed.
```

Nodes

`Variables` $(v : \texttt{Type})\ (mono : \mathrm{monoid}\ v)$.

Nodes

```
Variables (v : Type) (mono : monoid v).

Variables (A : Type) (measure : A → v).

Inductive node : Type :=
| Node2 : ∀ x y, { s : v | s = ‖ x ‖ · ‖ y ‖ } → node
| Node3 : ∀ x y z, { s : v | s = ‖ x ‖ · ‖ y ‖ · ‖ z ‖ } → node.
```

Nodes

```
Variables (v : Type) (mono : monoid v).

Variables (A : Type) (measure : A → v).

Inductive node : Type :=
| Node2 : ∀ x y, { s : v | s = ‖ x ‖ · ‖ y ‖ } → node
| Node3 : ∀ x y z, { s : v | s = ‖ x ‖ · ‖ y ‖ · ‖ z ‖ } → node.

Program Definition node2 (x y : A) : node :=
  Node2 x y (‖ x ‖ · ‖ y ‖).

Program Definition node_measure (n : node) : v :=
  match n with Node2 _ _ s ⇒ s | Node3 _ _ _ s ⇒ s end.
```

Dependent Finger Trees

```
Inductive fingertree (A : Type)  : Type :=
| Empty : fingertree A
| Single : ∀ x : A, fingertree A
| Deep : ∀ (l : digit A) (m : v),
   fingertree (node A)  →
   ∀ (r : digit A),
   fingertree A.
```

```
node : ∀ (A : Type) (me : A → v), Type
```

Dependent Finger Trees

`Inductive` fingertree (A : `Type`) ($me : A \to v$) : `Type` :=

| Empty : fingertree A me

| Single : $\forall\, x : A$, fingertree A me

| Deep : $\forall\, (l : \text{digit } A)\ (m : v)$,
fingertree (node A me) ($node\_measure\ A\ me$) $\to$
$\forall\, (r : \text{digit } A)$,
fingertree A me.

$$node : \forall\, (A : \texttt{Type})\ (me : A \to v),\ \texttt{Type}$$

$$node\_measure\ A\ (me : A \to v) : node\ A\ measure \to v$$

Dependent Finger Trees

$\texttt{Inductive}\ \mathrm{fingertree}\ (A : \texttt{Type})\ (me : A \to v) : v \to \texttt{Type} :=$

$\mid \mathrm{Empty} : \mathrm{fingertree}\ A\ me\ \varepsilon$

$\mid \mathrm{Single} : \forall\ x : A,\ \mathrm{fingertree}\ A\ me\ (me\ x)$

$\mid \mathrm{Deep} : \forall\ (l : \mathrm{digit}\ A)\ (m : v),$
$\quad \mathrm{fingertree}\ (\mathrm{node}\ A\ me)\ (node\_measure\ A\ me)\ m \to$
$\quad \forall\ (r : \mathrm{digit}\ A),$
$\quad \mathrm{fingertree}\ A\ me$
$\qquad (digit\_measure\ me\ l \cdot m \cdot digit\_measure\ me\ r).$

Adding to the left

`Program Fixpoint` $\mathit{add\_left}\ A\ (\mathit{me} : A \rightarrow v)$
$(a : A)\ (s : v)\ (t : \mathsf{fingertree}\ \mathit{me}\ s)$ `{struct` t`}` :
$\mathsf{fingertree}\ \mathit{me}\ (\mathit{me}\ a \cdot s)$:=

Adding to the left

```
Program Fixpoint add_left A (me : A → v)
  (a : A) (s : v) (t : fingertree me s) {struct t} :
  fingertree me (me a · s) :=
  match t with
    | Empty ⇒ Single a ← measure a = measure a · ε
    | Single b ⇒ Deep (One a) Empty (One b)
    | Deep pr st' t' sf ⇒
       . . .
  end.
```

Adding to the left

```
Program Fixpoint add_left A (me : A → v)
  (a : A) (s : v) (t : fingertree me s) {struct t} :
  fingertree me (me a · s) :=
  match t with
    | Empty ⇒ Single a ← measure a = measure a · ε
    | Single b ⇒ Deep (One a) Empty (One b)
    | Deep pr st' t' sf ⇒
       match pr with
         | Four b c d e ⇒
            let sub := add_left (node3 me c d e) t' in
              Deep (Two a b) sub sf
         | x ⇒ Deep (add_digit_left a pr) t' sf
      end
  end.
```

A glimpse of Finger Trees in COQ

The development

- Certified implementation of Finger Trees, sequences and ropes built on top of Finger Trees.
- ∼ 1200 lines of specification, ∼ 1400 of proof, mostly unchanged code.

	HASKELL	PROGRAM		
	Lines	L.o.C.	Obls	L.o.P.
app	200	200	100	auto
split	20	30	14	200
FingerTree	650	600	n.a.	400

A glimpse of Finger Trees in COQ

The development

- Certified implementation of Finger Trees, sequences and ropes built on top of Finger Trees.
- $\sim$ 1200 lines of specification, $\sim$ 1400 of proof, mostly unchanged code.
- Extracts to HASKELL and OCAML (with magic).

Module version fast enough for the ICFP contest. DEMO!

Experiment conclusions

- \+ PROGRAM scales ;
- \+ Subset types arise naturally ;
- \+ Dependent types are a powerful specification tool ;
- – Need more language technology, e.g: overloading ;
- – Some difficulties with reasoning and computing.

Conclusion

Our contributions

- A more flexible programming language, (almost) conservative over CIC, integrated with the existing environment and a formal justification of "*Predicate subtyping*".
- A tool to make programming in COQ using the full language possible, which can effectively be used for non-trivial developments.

Ongoing and future work

- Reasoning support through tactics
- Implementation of proof-irrelevance in COQ's kernel
- Overloading support through a typeclass mechanism.

The End

http://www.lri.fr/~sozeau/research/russell.en.html