Programing in Coq

Matthieu Sozeau
under the direction of Christine Paulin-Mohring

LRI, Univ. Paris-Sud - Démons Team & INRIA Futurs - ProVal Project

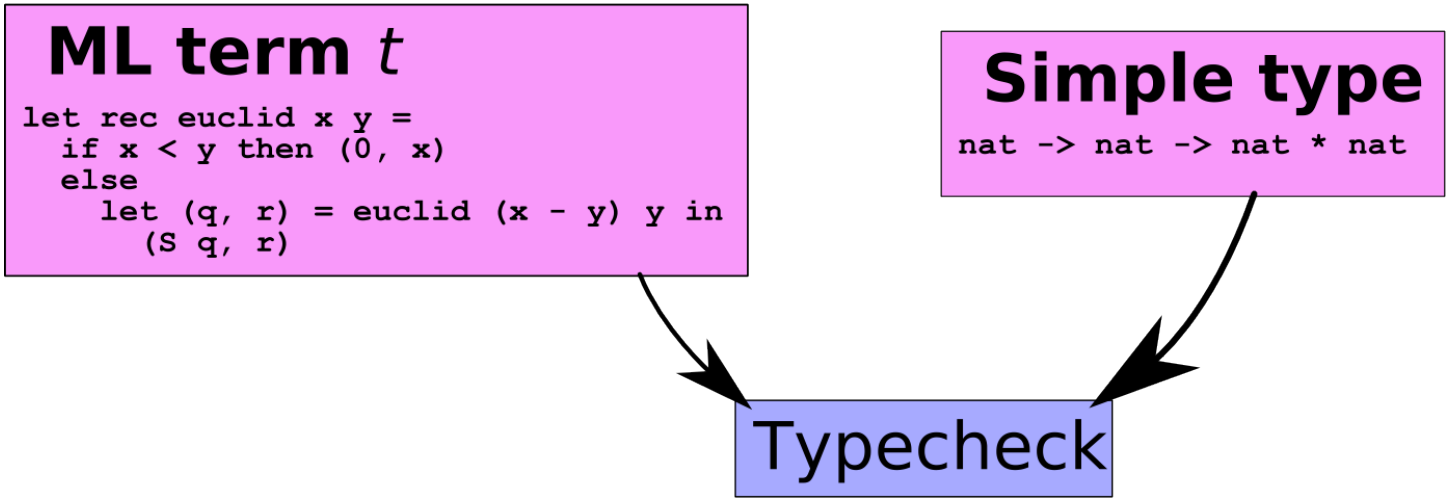
Démons-ProVal-LogiCal seminar
9 march 2007

The Big Picture

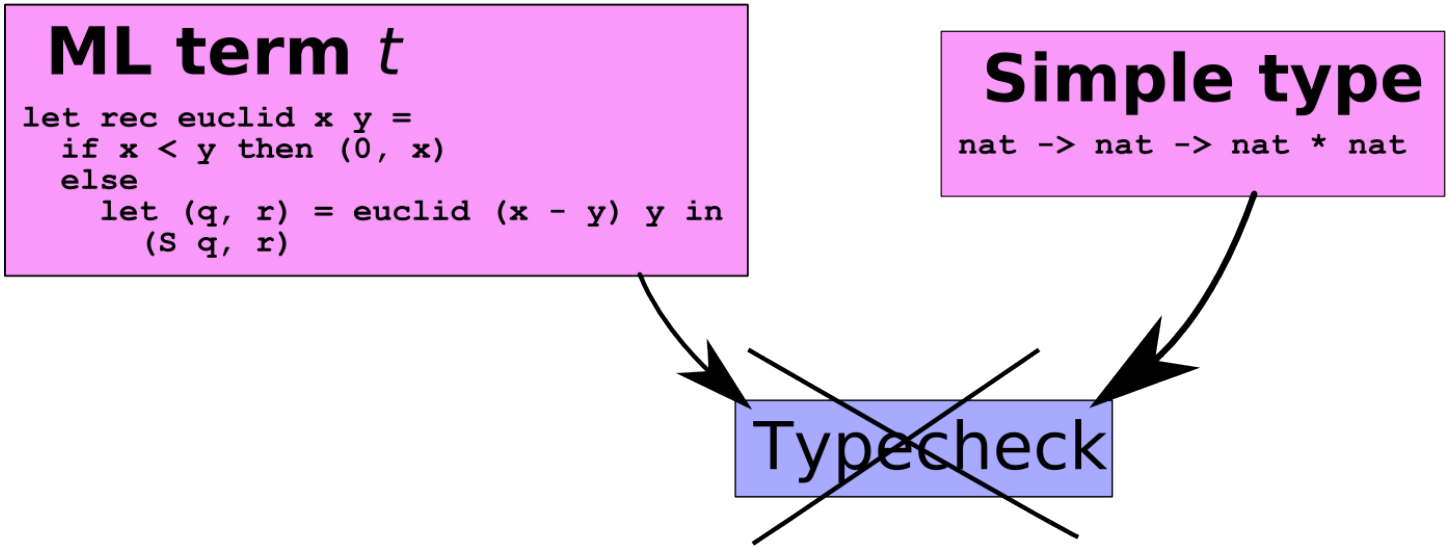
ML term *t*

```
let rec euclid x y =
  if x < y then (0, x)
  else
    let (q, r) = euclid (x - y) y in
      (S q, r)
```

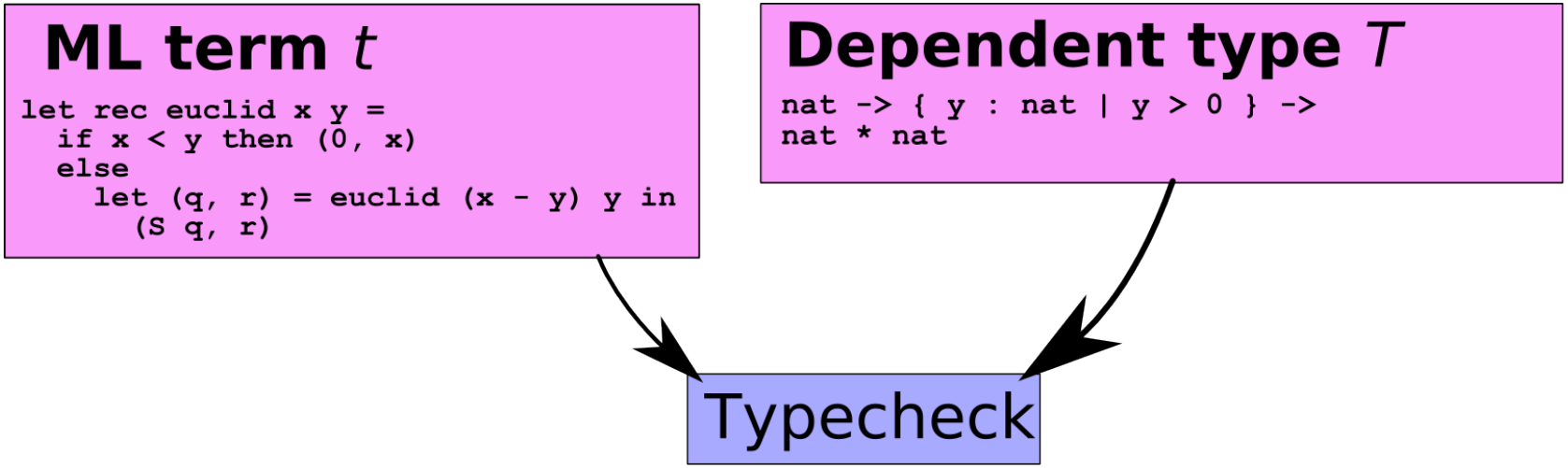
The Big Picture



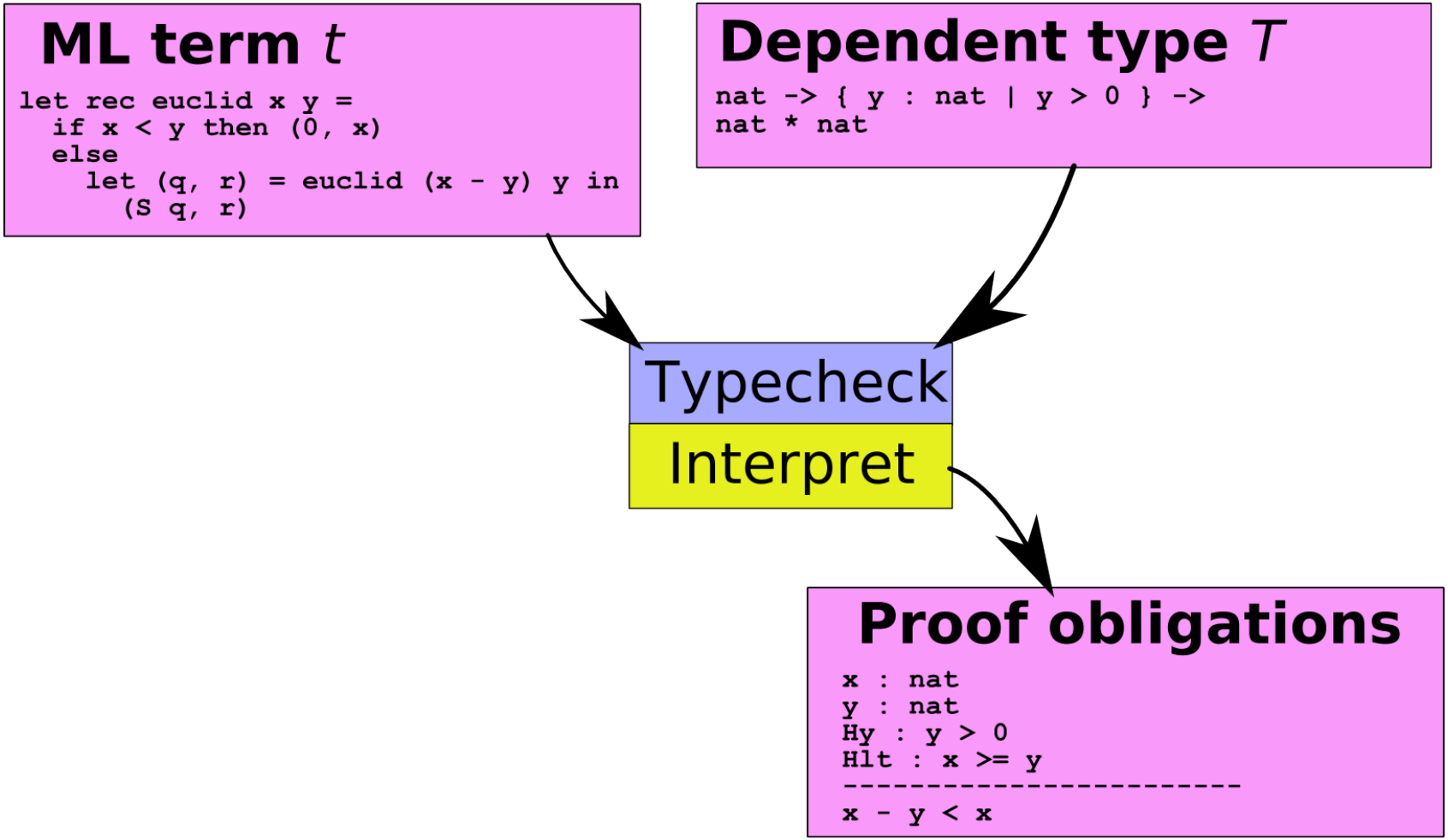
The Big Picture



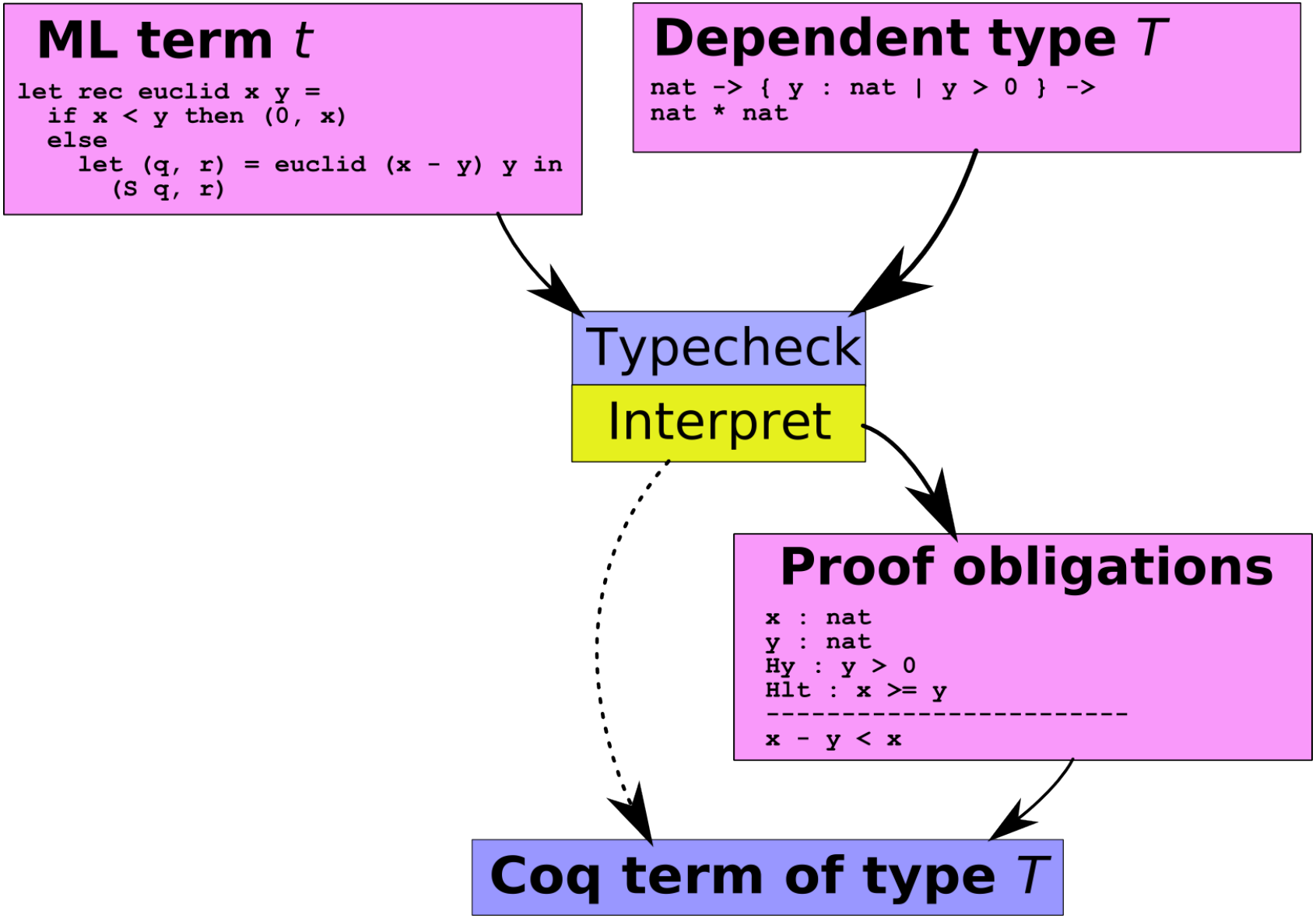
The Big Picture



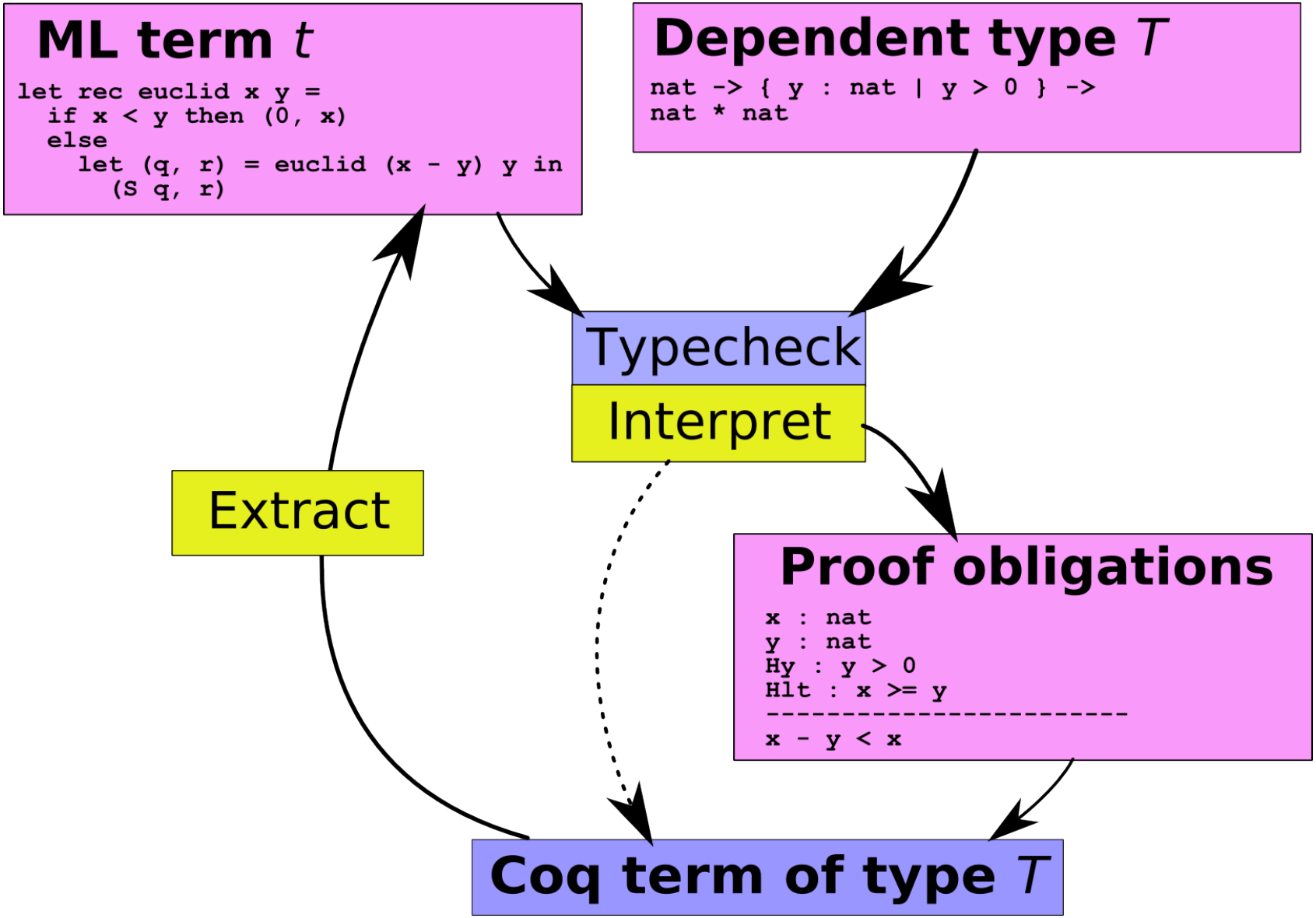
The Big Picture



The Big Picture



The Big Picture

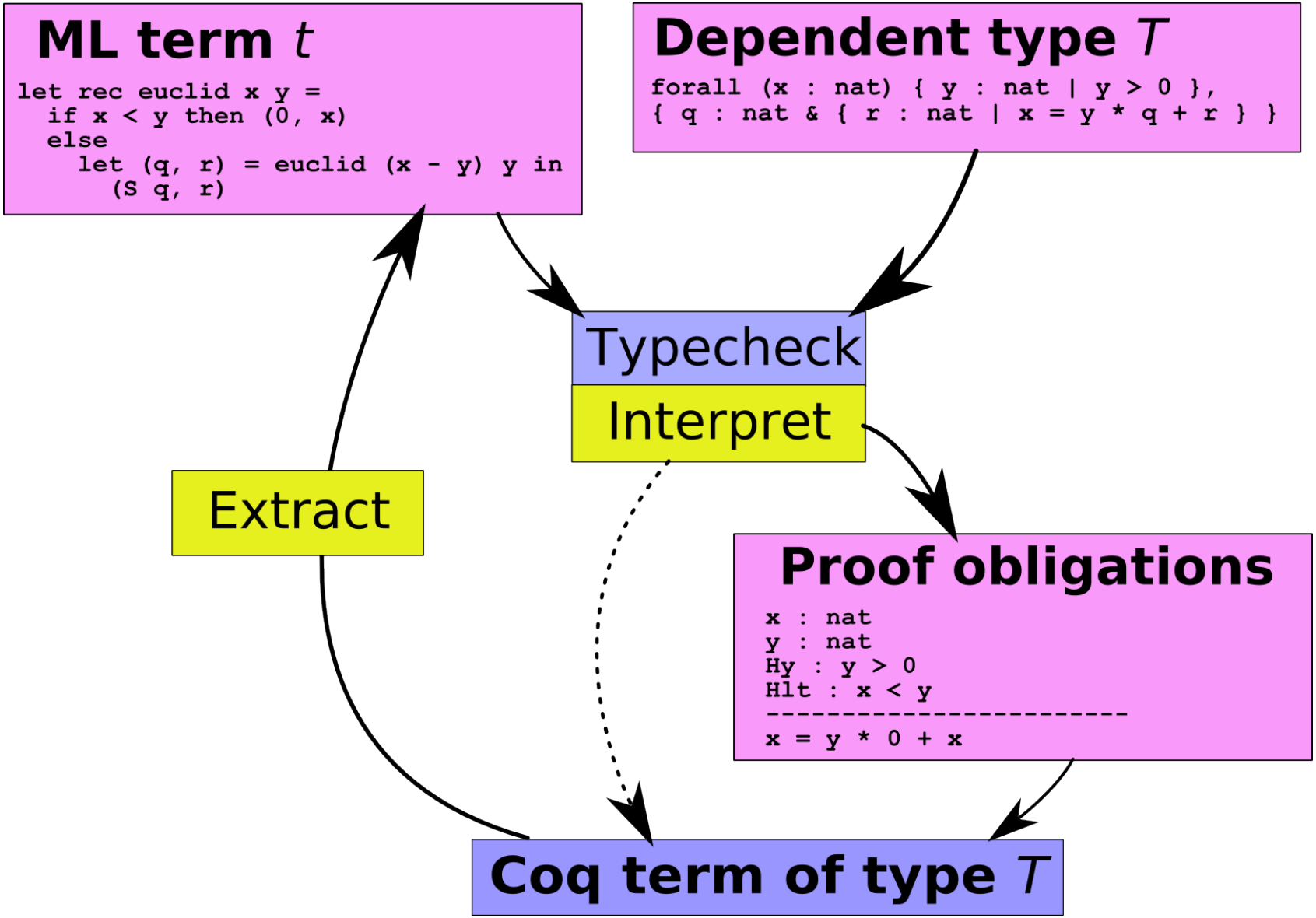


The Big Picture

```
Inductive diveucl a b : Set :=
  divex : ∀ q r, b > r → a = q × b + r → diveucl a b.

Lemma eucl_dev : ∀ n, n > 0 → ∀ m:nat, diveucl m n.
Proof.
  intros b H a; pattern a in ⊢ ×; apply gt_wf_rec; intros n
H0.
  elim (le_gt_dec b n).
  intro lebn.
  elim (H0 (n - b)); auto with arith.
  intros q r g e.
  apply divex with (S q) r; simpl in ⊢ ×; auto with arith.
  elim plus_assoc.
  elim e; auto with arith.
  intros gtbn.
  apply divex with 0 n; simpl in ⊢ ×; auto with arith.
Qed.
```

The Big Picture



A simple idea

Definition

$\{x : T \mid P\}$ is the set of objects of set T verifying property P.

- Useful for specifying, widely used in mathematics ;
- Links object and property.

A simple idea

Definition

$\{x : T \mid P\}$ is the set of objects of set T verifying property P.

- Useful for specifying, widely used in mathematics ;
- Links object and property.

Adapting the idea

$$\frac{t : T \qquad P[t/x]}{t \quad : \{ x : T \mid P \}} \qquad \frac{t : \{ x : T \mid P \}}{t : T}$$

A simple idea

Definition

$\{x : T \mid P\}$ is the set of objects of set T verifying property P.

- Useful for specifying, widely used in mathematics ;
- Links object and property.

Adapting the idea

$$\frac{t : T \qquad p : P[t/x]}{(t, p) : \{\, x : T \mid P \,\}} \qquad \frac{t : \{\, x : T \mid P \,\}}{\mathtt{proj}\ t : T}$$

From "*Predicate subtyping*"...

PVS

- Specialized typing algorithm for subset types, generating *Type-checking conditions*.

$t : \{ x : T \mid P \}$	used as	$t : T$	ok
$t : T$	used as	$t : \{ x : T \mid P \}$	if $P[t/x]$

From "*Predicate subtyping*"...

PVS

- Specialized typing algorithm for subset types, generating *Type-checking conditions*.

$t : \{ x : T \mid P \}$	used as	$t : T$	ok
$t : T$	used as	$t : \{ x : T \mid P \}$	if $P[t/x]$

\+ Practical success ;

From "*Predicate subtyping*"...

PVS

- Specialized typing algorithm for subset types, generating *Type-checking conditions*.

$t : \{ x : T \mid P \}$	used as	$t : T$	ok
$t : T$	used as	$t : \{ x : T \mid P \}$	if $P[t/x]$

+ Practical success ;

− No strong safety guarantee in PVS.

...to Subset coercions

1. A property-irrelevant language (RUSSELL) with decidable typing ;

$$\frac{\Gamma \vdash t : \{\, x : T \mid P \,\}}{\Gamma \vdash \quad t : T}$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma, x : T \vdash P : \mathtt{Prop}}{\Gamma \vdash \; t \; : \{\, x : T \mid P \,\}}$$

...to Subset coercions

1. A property-irrelevant language (RUSSELL) with decidable typing ;
2. A total traduction to COQ terms with holes ;

$$\frac{\Gamma \vdash t : \{\, x : T \mid P \,\}}{\Gamma \vdash \mathtt{proj}\ t : T}$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma, x : T \vdash P : \mathtt{Prop}}{\Gamma \vdash (t, ?) : \{\, x : T \mid P \,\}}\ \Gamma \vdash ? : P[t/x]$$

...to Subset coercions

1. A property-irrelevant language (RUSSELL) with decidable typing ;
2. A total traduction to COQ terms with holes ;
3. A mechanism to turn the holes into proof obligations and manage them.

$$\frac{\Gamma \vdash t : \{\, x : T \mid P \,\}}{\Gamma \vdash \mathtt{proj}\ t : T}$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma, x : T \vdash P : \mathtt{Prop} \qquad \Gamma \vdash p : P[t/x]}{\Gamma \vdash (t, p) : \{\, x : T \mid P \,\}}$$

Russell syntax

$$
\begin{array}{lcl}
x & \in & \mathcal{V} \\
\\
s, t, u, v & ::= & x \\
 & | & \texttt{Set} \\
 & | & \texttt{Prop} \\
 & | & \texttt{Type}
\end{array}
$$

Russell syntax

$$
\begin{array}{lcl}
x & \in & \mathcal{V} \\
\\
s, t, u, v & ::= & x \\
 & | & \texttt{Set} \\
 & | & \texttt{Prop} \\
 & | & \texttt{Type} \\
 & | & \lambda x : s.t \\
 & | & s\ t \\
 & | & \Pi x : s.t
\end{array}
$$

Russell syntax

$$
\begin{array}{lcl}
x & \in & \mathcal{V} \\
\\
s, t, u, v & ::= & x \\
 & | & \texttt{Set} \\
 & | & \texttt{Prop} \\
 & | & \texttt{Type} \\
 & | & \lambda x : s.t \\
 & | & s\ t \\
 & | & \Pi x : s.t \\
 & | & (u, v)_{\Sigma x:s.t} \\
 & | & \pi_1\ s \mid \pi_2\ s \\
 & | & \Sigma x : s.t
\end{array}
$$

Russell syntax

$$
\begin{array}{lcl}
x & \in & \mathcal{V} \\
\\
s, t, u, v & ::= & x \\
 & | & \texttt{Set} \\
 & | & \texttt{Prop} \\
 & | & \texttt{Type} \\
 & | & \lambda x : s.t \\
 & | & s\ t \\
 & | & \Pi x : s.t \\
 & | & (u, v)_{\Sigma x:s.t} \\
 & | & \pi_1\ s \mid \pi_2\ s \\
 & | & \Sigma x : s.t \\
 & | & \{\ x : s \mid t\ \}
\end{array}
$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \equiv_{\beta\pi} T : s}{\Gamma \vdash t : T}$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \rhd T : s}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash T \equiv_{\beta\pi} U : s}{\Gamma \vdash T \rhd U : s}$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \rhd T : s}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash T \equiv_{\beta\pi} U : s}{\Gamma \vdash T \rhd U : s}$$

$$\frac{\Gamma \vdash S \rhd T : s \qquad \Gamma \vdash T \rhd U : s}{\Gamma \vdash S \rhd U : s}$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \rhd T : s}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash T \equiv_{\beta\pi} U : s}{\Gamma \vdash T \rhd U : s}$$

$$\frac{\Gamma \vdash S \rhd T : s \qquad \Gamma \vdash T \rhd U : s}{\Gamma \vdash S \rhd U : s}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : U \vdash P : \mathtt{Prop}}{\Gamma \vdash \{\, x : U \mid P \,\} \rhd V : \mathtt{Set}}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : V \vdash P : \mathtt{Prop}}{\Gamma \vdash U \rhd \{\, x : V \mid P \,\} : \mathtt{Set}}$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \rhd T : s}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash T \equiv_{\beta\pi} U : s}{\Gamma \vdash T \rhd U : s}$$

$$\frac{\Gamma \vdash S \rhd T : s \qquad \Gamma \vdash T \rhd U : s}{\Gamma \vdash S \rhd U : s}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : U \vdash P : \mathtt{Prop}}{\Gamma \vdash \{\, x : U \mid P \,\} \rhd V : \mathtt{Set}}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : V \vdash P : \mathtt{Prop}}{\Gamma \vdash U \rhd \{\, x : V \mid P \,\} : \mathtt{Set}}$$

$$\frac{\Gamma \vdash 0 : \mathbb{N} \qquad \Gamma \vdash \mathbb{N} \rhd \{\, x : \mathbb{N} \mid x \neq 0 \,\} : \mathtt{Set}}{\Gamma \vdash 0 : \{\, x : \mathbb{N} \mid x \neq 0 \,\}}$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \rhd T : s}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash T \equiv_{\beta\pi} U : s}{\Gamma \vdash T \rhd U : s}$$

$$\frac{\Gamma \vdash S \rhd T : s \qquad \Gamma \vdash T \rhd U : s}{\Gamma \vdash S \rhd U : s}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : U \vdash P : \mathtt{Prop}}{\Gamma \vdash \{\, x : U \mid P \,\} \rhd V : \mathtt{Set}}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : V \vdash P : \mathtt{Prop}}{\Gamma \vdash U \rhd \{\, x : V \mid P \,\} : \mathtt{Set}}$$

$$\frac{\dfrac{\Gamma \vdash 0 : \mathbb{N} \qquad \Gamma \vdash \mathbb{N} \rhd \{\, x : \mathbb{N} \mid x \neq 0 \,\} : \mathtt{Set}}{\Gamma \vdash 0 : \{\, x : \mathbb{N} \mid x \neq 0 \,\}}}{\Gamma \vdash ? : 0 \neq 0}$$

RUSSELL typing $\vdash$ and coercion $\rhd$

Calculus of Constructions with

$$\frac{\Gamma \vdash t : U \qquad \Gamma \vdash U \rhd T : s}{\Gamma \vdash t : T} \qquad \frac{\Gamma \vdash T \equiv_{\beta\pi} U : s}{\Gamma \vdash T \rhd U : s}$$

$$\frac{\Gamma \vdash S \rhd T : s \qquad \Gamma \vdash T \rhd U : s}{\Gamma \vdash S \rhd U : s}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : U \vdash P : \mathtt{Prop}}{\Gamma \vdash \{\, x : U \mid P \,\} \rhd V : \mathtt{Set}}$$

$$\frac{\Gamma \vdash U \rhd V : \mathtt{Set} \qquad \Gamma, x : V \vdash P : \mathtt{Prop}}{\Gamma \vdash U \rhd \{\, x : V \mid P \,\} : \mathtt{Set}}$$

$$\frac{\Gamma \vdash U \rhd T : s_1 \qquad \Gamma, x : U \vdash V \rhd W : s_2}{\Gamma \vdash \Pi x : T.V \rhd \Pi x : U.W : s_2}$$

Results

Theorem (Decidability of type checking and type inference)

$\Gamma \vdash t : T$ *is decidable.*

$$\textsc{App}\ \frac{\Gamma \vdash f : T \quad \Gamma \vdash T \rhd \Pi x : A.B : s \qquad \Gamma \vdash e : E \qquad \Gamma \vdash E \rhd A : s'}{\Gamma \vdash (f\ e) : B[e/x]}$$

Results

Theorem (Decidability of type checking and type inference)

$\Gamma \vdash t : T$ *is decidable.*

Lemma (Elimination of transitivity)

If $T \rhd U \wedge U \rhd V$ *then* $T \rhd V$.

Theorem (Subject Reduction)

If $\Gamma \vdash t : T$ *and* $t \to_{\beta\pi} t'$ *then* $\Gamma \vdash t' : T$.

$$\textsc{App}\ \frac{\Gamma \vdash f : T \quad \Gamma \vdash T \rhd \Pi x : A.B : s \qquad \Gamma \vdash e : E \qquad \Gamma \vdash E \rhd A : s'}{\Gamma \vdash (f\ e) : B[e/x]}$$

From Russell to Coq

The target system : Cic with metavariables

$$\frac{\Gamma \vdash_? t : T \qquad \Gamma \vdash_? p : P[t/x]}{\Gamma \vdash_? \mathsf{elt}\ T\ P\ t\ p : \{\ x : T \mid P\ \}}$$

$$\frac{\Gamma \vdash_? t : \{\ x : T \mid P\ \}}{\Gamma \vdash_? \sigma_1\ t : T} \qquad \frac{\Gamma \vdash_? t : \{\ x : T \mid P\ \}}{\Gamma \vdash_? \sigma_2\ t : P[\sigma_1\ t/x]}$$

$$\frac{\Gamma \vdash_? P : \mathtt{Prop}}{\Gamma \vdash_? ?_P : P}$$

We build an interpretation $[\![\_]\!]_\Gamma$ from Russell to Cic$_?$ terms.

From Russell to Coq

The target system : Cic with metavariables

$$\frac{\Gamma \vdash_? t : T \qquad \Gamma \vdash_? p : P[t/x]}{\Gamma \vdash_? \mathsf{elt}\ T\ P\ t\ p : \{\ x : T \mid P\ \}}$$

$$\frac{\Gamma \vdash_? t : \{\ x : T \mid P\ \}}{\Gamma \vdash_? \sigma_1\ t : T} \qquad \frac{\Gamma \vdash_? t : \{\ x : T \mid P\ \}}{\Gamma \vdash_? \sigma_2\ t : P[\sigma_1\ t/x]}$$

$$\frac{\Gamma \vdash_? P : \mathtt{Prop}}{\Gamma \vdash_? ?_P : P}$$

We build an interpretation $[\![\_]\!]_\Gamma$ from Russell to Cic$_?$ terms.

Our goal

If $\Gamma \vdash t : T$ then $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$.

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies $[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma$.

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies

$$[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma.$$

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \quad : T \rhd U}$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies

$$[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma.$$

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies $[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma$.

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \qquad : \{\, x : T \mid P \,\} \rhd T$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies

$$[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma.$$

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \sigma_1 \bullet : \{ x : T \mid P \} \rhd T$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies

$$[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma.$$

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \sigma_1 \bullet : \{ x : T \mid P \} \rhd T$$

$$\Gamma \vdash_? \qquad\qquad : T \rhd \{ x : T \mid P \}$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies $[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma$.

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \sigma_1 \bullet : \{ x : T \mid P \} \rhd T$$

$$\Gamma \vdash_? \mathit{elt}\ \_\ \_\ \bullet\ ?_{[\![P]\!]_{\Gamma, x:T}[\bullet/x]} : T \rhd \{ x : T \mid P \}$$

Deriving explicit coercions

Interpretation of coercions

If $\Gamma \vdash T \rhd U : s$ then $\Gamma \vdash_? c[\bullet] : T \rhd U$ which implies $[\![\Gamma]\!], x : [\![T]\!]_\Gamma \vdash_? c[x] : [\![U]\!]_\Gamma$.

Definition

$$\frac{T \equiv_{\beta\pi} U}{\Gamma \vdash_? \bullet : T \rhd U}$$

$$\Gamma \vdash_? \sigma_1 \bullet : \{ x : T \mid P \} \rhd T$$

$$\Gamma \vdash_? \mathit{elt}\ \_\ \_\ \bullet\ ?_{[\![P]\!]_{\Gamma, x:T}[\bullet/x]} : T \rhd \{ x : T \mid P \}$$

Example

$$\frac{\Gamma \vdash_? 0 : \mathbb{N} \qquad \Gamma \vdash_? \mathit{elt}\ \_\ \_\ \bullet\ ?_{(x \neq 0)[\bullet/x]} : \mathbb{N} \rhd \{ x : \mathbb{N} \mid x \neq 0 \}}{\Gamma \vdash_? \mathit{elt}\ \_\ \_\ 0\ ?_{0 \neq 0} : \{ x : \mathbb{N} \mid x \neq 0 \}}$$

Interpretation of terms

Example (Application)

$$\frac{\Gamma \vdash f : T \quad \Gamma \vdash T \rhd \Pi x : V.W : s \quad \Gamma \vdash u : U \quad \Gamma \vdash U \rhd V : s'}{\Gamma \vdash (f\ u) : W[u/x]}$$

$$\begin{aligned} [\![f\ u]\!]_\Gamma \quad = \quad & \textit{let}\ \pi = \textit{coerce}_\Gamma\ T\ (\Pi x : V.W)\ \textit{in} \\ & \textit{let}\ c = \textit{coerce}_\Gamma\ U\ V\ \textit{in} \\ & (\pi[[\![f]\!]_\Gamma])\ (c[[\![u]\!]_\Gamma]) \end{aligned}$$

Theorem (Soundness)

If $\Gamma \vdash t : T$ *then* $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$.

Theoretical matters ...

$\vdash_?$'s equational theory:

(β)	$(\lambda x : X.e)\ v$	$\equiv$	$e[v/x]$	
(π_i)	$\pi_i\ (e_1, e_2)_T$	$\equiv$	e_i	
(σ_i)	$\sigma_i\ (\mathsf{elt}\ E\ P\ e_1\ e_2)$	$\equiv$	e_i	
(η)	$(\lambda x : X.e\ x)$	$\equiv$	e	if $x \notin FV(e)$
(SP)	$\mathsf{elt}\ E\ P\ (\sigma_1\ e)\ (\sigma_2\ e)$	$\equiv$	e	

Theoretical matters ...

$\vdash_?$'s equational theory:

(β)	$(\lambda x : X.e)\ v$	$\equiv$	$e[v/x]$	
(π_i)	$\pi_i\ (e_1, e_2)_T$	$\equiv$	e_i	
(σ_i)	$\sigma_i\ (\mathsf{elt}\ E\ P\ e_1\ e_2)$	$\equiv$	e_i	
(η)	$(\lambda x : X.e\ x)$	$\equiv$	e	if $x \notin FV(e)$
(SP)	$\mathsf{elt}\ E\ P\ (\sigma_1\ e)\ (\sigma_2\ e)$	$\equiv$	e	
(σ)	$\mathsf{elt}\ E\ P\ t\ p$	$\equiv$	$\mathsf{elt}\ E\ P\ t'\ p'$	if $t \equiv t'$

$\Rightarrow$ Proof Irrelevance

Theoretical matters ...

$\vdash_?$'s equational theory:

(β)	$(\lambda x : X.e)\ v$	$\equiv$	$e[v/x]$	
(π_i)	$\pi_i\ (e_1, e_2)_T$	$\equiv$	e_i	
(σ_i)	$\sigma_i\ (\mathsf{elt}\ E\ P\ e_1\ e_2)$	$\equiv$	e_i	
(η)	$(\lambda x : X.e\ x)$	$\equiv$	e	if $x \notin FV(e)$
(SP)	$\mathsf{elt}\ E\ P\ (\sigma_1\ e)\ (\sigma_2\ e)$	$\equiv$	e	
(σ)	$\mathsf{elt}\ E\ P\ t\ p$	$\equiv$	$\mathsf{elt}\ E\ P\ t'\ p'$	if $t \equiv t'$

$\Rightarrow$ Proof Irrelevance

...have practical effects

Difficulty to reason on code: $f(\mathsf{elt}\ T\ P\ x\ p_1) \not\equiv f(\mathsf{elt}\ T\ P\ x\ p_2)$.

Inductive types

A natural extension: Reindexing

Let $I\ \overrightarrow{x}$ be an inductive type with *real* arguments $\overrightarrow{x}$. We can coerce any object of this type to an object of $I\ \overrightarrow{y}$ provided $\overrightarrow{x} = \overrightarrow{y}$.

Inductive types

A natural extension: Reindexing

Let $I\ \overrightarrow{x}$ be an inductive type with *real* arguments $\overrightarrow{x}$. We can coerce any object of this type to an object of $I\ \overrightarrow{y}$ provided $\overrightarrow{x} = \overrightarrow{y}$.

$$\frac{\Gamma \vdash v : \mathtt{vec}\ (m + 0) \qquad \Gamma \vdash \mathtt{vec}\ (m + 0) \rhd \mathtt{vec}\ m : \mathtt{Set}}{\Gamma \vdash v : \mathtt{vec}\ m}$$

Inductive types

A natural extension: Reindexing

Let $I\ \overrightarrow{x}$ be an inductive type with *real* arguments $\overrightarrow{x}$. We can coerce any object of this type to an object of $I\ \overrightarrow{y}$ provided $\overrightarrow{x} = \overrightarrow{y}$.

$$\frac{\Gamma \vdash v : \mathtt{vec}\ (m+0) \qquad \Gamma \vdash \mathtt{vec}\ (m+0) \rhd \mathtt{vec}\ m : \mathtt{Set}}{\Gamma \vdash v : \mathtt{vec}\ m}$$

$$\dfrac{\ldots \qquad \dfrac{\Gamma \vdash_? ?_{m+0=m} : m+0=m}{\Gamma \vdash_? \mathtt{eq\_rec}\ \mathbb{N}\ (m+0)\ \mathtt{vec}\ \bullet\ m\ ?_{m+0=m} : \mathtt{vec}\ (m+0) \rhd \mathtt{vec}\ m}}{\Gamma \vdash_? \mathtt{eq\_rec}\ \mathbb{N}\ (m+0)\ \mathtt{vec}\ \ v\ m\ ?_{m+0=m} : \mathtt{vec}\ m}$$

$$\mathtt{eq\_rec} : \forall (A : \mathtt{Set})(x : A)(P : A \to \mathtt{Set}), P\ x \to \forall y, x = y \to P\ y$$

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the Coq parser.

```
Program Definition f : T := t .
```

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the Coq parser.
2. Typecheck $\Gamma \vdash t : T$ and generate $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$;

$$\texttt{Program Definition f} \; : [\![T]\!]_\Gamma := [\![t]\!]_\Gamma \qquad .$$

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the Coq parser.
2. Typecheck $\Gamma \vdash t : T$ and generate $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$;
3. Interactive proving of obligations ;

`Program Definition f` $: [\![T]\!]_\Gamma := [\![t]\!]_\Gamma +$ obligations.

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the Coq parser.
2. Typecheck $\Gamma \vdash t : T$ and generate $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$;
3. Interactive proving of obligations ;
4. Final definition.

`Definition f` $: [\![T]\!]_\Gamma := [\![t]\!]_\Gamma +$ obligations.

The Program vernacular

Architecture

Wrap around Coq's vernacular commands (`Definition`, `Fixpoint`, `Lemma`, ...).

1. Use the Coq parser.
2. Typecheck $\Gamma \vdash t : T$ and generate $[\![\Gamma]\!] \vdash_? [\![t]\!]_\Gamma : [\![T]\!]_\Gamma$;
3. Interactive proving of obligations ;
4. Final definition.

`Definition f` $: [\![T]\!]_\Gamma := [\![t]\!]_\Gamma +$ obligations.

Remark (Restriction)

We assume $\Gamma \vdash_{CCI} [\![T]\!]_\Gamma : s$.

Hello world: Euclidian division

DEMO

Pattern-matching revisited

From patterns to equations:

$$
\begin{array}{llll}
\textbf{match} & e & \textbf{return} & T \textbf{ with} \\
\mid p_1 & & \Rightarrow & t_1 \\
 & & \vdots & \\
\mid p_n & & \Rightarrow & t_n \\
\textbf{end} & & &
\end{array}
$$

where $\nu(e)$ coerces e to an inductive object.

Pattern-matching revisited

From patterns to equations:

$$
\begin{array}{lll}
\textbf{match}\ \nu(e)\ \textbf{as}\ t & \textbf{return} & t = \nu(e) \to T\ \textbf{with} \\
\mid p_1 & \Rightarrow & \textbf{fun}\ H : p_1 = \nu(e) \Rightarrow t_1 \\
 & \vdots & \\
\mid p_n & \Rightarrow & \textbf{fun}\ H : p_n = \nu(e) \Rightarrow t_n \\
\textbf{end} & & \\
(\textbf{refl\_equal}\ \nu(e)) & &
\end{array}
$$

where $\nu(e)$ coerces e to an inductive object.

Pattern-matching revisited

From patterns to equations:

$$
\begin{array}{lll}
\textbf{match } \nu(e) \textbf{ as } t & \textbf{return} & t = \nu(e) \to T \textbf{ with} \\
\mid p_1 & \Rightarrow & \textbf{fun } H : p_1 = \nu(e) \Rightarrow t_1 \\
 & \vdots & \\
\mid p_n & \Rightarrow & \textbf{fun } H : p_n = \nu(e) \Rightarrow t_n \\
\textbf{end} & & \\
(\textbf{refl\_equal } \nu(e)) & &
\end{array}
$$

where $\nu(e)$ coerces e to an inductive object.

Further refinements

- Generalized to dependent inductive types ;
- Each branch typed only once ;
- Add inequalities for intersecting patterns.

Sugar

Obligations

Unresolved implicits (_) are turned into obligations.

Bang

$! \triangleq (\texttt{False\_rect}\ \_\ \_)$ where

$\texttt{False\_rect} : \forall A : \texttt{Type}, \texttt{False} \to A$. It corresponds to ML's `assert(false)`.

Example

$\mathit{match}\ 0\ \mathit{with}\ 0 \Rightarrow 0 \mid \mathrm{n} \Rightarrow !\ \mathit{end}$

Recursion

Support for well-founded recursion (and measures).

$$\texttt{Program Fixpoint}\ f\ (a : nat)\ \{\mathbf{wf}\ lt\ a\} : \mathbb{N} := b$$

Recursion

Support for well-founded recursion (and measures).

$$\texttt{Program Fixpoint}\ f\ (a : nat)\ \{\mathbf{wf}\ lt\ a\} : \mathbb{N} := b$$

$$\begin{array}{lcl} a & : & \mathbb{N} \\ f & : & \{x : \mathbb{N} \mid x < a\} \to \mathbb{N} \\ \hline b & : & \mathbb{N} \end{array}$$

Digits

```
Section Digit.
  Variable A : Type.

  Inductive digit : Type :=
  | One : A → digit
  | Two : A → A → digit
  | Three : A → A → A → digit
  | Four : A → A → A → A → digit.

  Definition full x := match x with Four _ _ _ _ ⇒ True | _ ⇒ False end.
```

Digits

```
Section Digit.
  Variable A : Type.

  Inductive digit : Type :=
  | One : A → digit
  | Two : A → A → digit
  | Three : A → A → A → digit
  | Four : A → A → A → A → digit.

  Definition full x := match x with Four _ _ _ _ ⇒ True | _ ⇒ False end.

  Program Definition add_digit_left (a : A) (d : digit | ¬ full d) : digit :=
    match d with
      | One x ⇒ Two a x
      | Two x y ⇒ Three a x y
      | Three x y z ⇒ Four a x y z
      | Four _ _ _ _ ⇒ !
    end.
  Next Obligation.
    intros ; simpl in n ; auto.
  Qed.
```

Monoids & Nodes

```
Variable v : Type.
Variable mono : monoid v.

Notation "'ε'" := mempty mono.
Infix "·" := mappend mono (right associativity, at level 20).

Program Definition listMonoid (A : Type) :=
  @mkMonoid (list A) nil (@app A) _ _ _.
```

Monoids & Nodes

```
Variable v : Type.
Variable mono : monoid v.

Notation "'ε'" := mempty mono.
Infix "·" := mappend mono (right associativity, at level 20).

Program Definition listMonoid (A : Type) :=
  @mkMonoid (list A) nil (@app A) _ _ _.

Section Nodes.
  Variable A : Type.
  Variable measure : A → v.
  Notation "'‖' x '‖'" := (measure x).

  Inductive node : Type :=
  | Node2 : ∀ x y, { s : v | s = ‖ x ‖ · ‖ y ‖ } → node
  | Node3 : ∀ x y z, { s : v | s = ‖ x ‖ · ‖ y ‖ · ‖ z ‖ } → node.
```

Monoids & Nodes

```
Variable v : Type.
Variable mono : monoid v.

Notation "'ε'" := mempty mono.
Infix "·" := mappend mono (right associativity, at level 20).

Program Definition listMonoid (A : Type) :=
  @mkMonoid (list A) nil (@app A) _ _ _.

Section Nodes.
  Variable A : Type.
  Variable measure : A → v.
  Notation "'‖' x '‖'" := (measure x).

  Inductive node : Type :=
  | Node2 : ∀ x y, { s : v | s = ‖ x ‖ · ‖ y ‖ } → node
  | Node3 : ∀ x y z, { s : v | s = ‖ x ‖ · ‖ y ‖ · ‖ z ‖ } → node.

  Program Definition node2 (x y : A) : node :=
    Node2 x y (‖ x ‖ · ‖ y ‖).

  Program Definition node_measure (n : node) : v :=
    match n with | Node2 _ _ s ⇒ s | Node3 _ _ _ s ⇒ s end.
```

Dependent Finger Trees

```
Inductive fingertree (A : Type) (measure : A → v) : v → Type :=
| Empty : fingertree measure ε
| Single : ∀ x : A, fingertree measure (measure x)
| Deep : ∀ (l : digit A) (m : v),
  @fingertree (node measure) nodeMeasure m → ∀ r : digit A,
  fingertree measure (digit_measure measure l · m · digit_measure measure r).
```

Dependent Finger Trees

```
Inductive fingertree (A : Type) (measure : A → v) : v → Type :=
| Empty : fingertree measure ε
| Single : ∀ x : A, fingertree measure (measure x)
| Deep : ∀ (l : digit A) (m : v),
  @fingertree (node measure) nodeMeasure m → ∀ r : digit A,
  fingertree measure (digit_measure measure l · m · digit_measure measure r).

  Program Fixpoint add_left A (measure : A → v) (a : A) (s : v)
    (t : fingertree measure s) {struct t} :
    fingertree measure (measure a · s) :=
    match t with
      | Empty ⇒ Single a
      | Single b ⇒ Deep (One a) Empty (One b)
      | Deep pr st' t' sf ⇒
        match pr with
          | Four b c d e ⇒ let sub := add_left (node3 measure c d e) t' in
            Deep (Two a b) sub sf
          | x ⇒ Deep (add_digit_left a pr) t' sf
        end
    end.
```

Dependent Finger Trees - cont'd

Two hundred lines, one hundred obligations concatenation

```
Definition app (A : Type) (measure : A → v)
  (xs : v) (x : fingertree measure xs)
  (ys : v) (y : fingertree measure ys) :
  fingertree measure (xs · ys).
```

Splitting, views can be defined in a similar way.

A glimpse of Finger Trees in COQ

The development

- Certified implementation of Finger Trees. Certified implementation of sequences built on top of Finger Trees.
- $\sim$ 1200 lines of specification, $\sim$ 1400 of proof, mostly unchanged code.
- Extracts to HASKELL and ML (uses dependent records).

A glimpse of Finger Trees in Coq

The development

- Certified implementation of Finger Trees. Certified implementation of sequences built on top of Finger Trees.
- ~ 1200 lines of specification, ~ 1400 of proof, mostly unchanged code.
- Extracts to Haskell and ML (uses dependent records).

Conclusions

- Program scales ;
- Need more language technology, e.g: overloading ;
- Subset types arise naturally ;
- Dependent types are a powerfull specification tool ;
- Some difficulties with reasonning and computing.

Conclusion

Our contributions

- A more flexible programming language, (almost) conservative over CIC, integrated with the existing environment and a formal justification of "*Predicate subtyping*".
- A tool to make programming in COQ using the full language possible, which can effectively be used for non-trivial developments.

Future work

Reasonning support through tactics, implementation of proof-irrelevance in COQ's kernel.

The End

http://www.lri.fr/~sozeau/research/russell.en.html

Why eta rules and proof irrelevance ?

Let $A, B : \mathtt{Type}$, $c : A \rhd B$, $d : B \rhd A$.

Why eta rules and proof irrelevance ?

Let $A, B : \texttt{Type}$, $c : A \rhd B$, $d : B \rhd A$.

Let $P : A \to \texttt{Prop}$ with $p : \Pi x : A, P\ x$ and $q : \Pi x : B, P\ x$.

Why eta rules and proof irrelevance ?

Let $A, B : \texttt{Type}$, $c : A \rhd B$, $d : B \rhd A$.

Let $P : A \to \texttt{Prop}$ with $p : \Pi x : A, P\ x$ and $q : \Pi x : B, P\ x$.

Consider: $p\ x =_{(P\ x)} q\ x$ in context $\Gamma \triangleq x : A$.

By $[\![\_]\!]_\Gamma$: $[\![P]\!]_\Gamma \triangleq P$, $[\![p]\!]_\Gamma \triangleq p$ and $[\![q]\!]_\Gamma \triangleq q : \Pi x : B, P\ d[x]$,

Why eta rules and proof irrelevance ?

Let $A, B : \texttt{Type}$, $c : A \rhd B$, $d : B \rhd A$.
Let $P : A \to \texttt{Prop}$ with $p : \Pi x : A, P\ x$ and $q : \Pi x : B, P\ x$.
Consider: $p\ x =_{(P\ x)} q\ x$ in context $\Gamma \triangleq x : A$.
By $[\![\_]\!]_\Gamma$: $[\![P]\!]_\Gamma \triangleq P$, $[\![p]\!]_\Gamma \triangleq p$ and $[\![q]\!]_\Gamma \triangleq q : \Pi x : B, P\ d[x]$,
hence: $[\![p\ x =_{(P\ x)} q\ x]\!]_\Gamma \triangleq p\ x =_{(P\ x)} q\ c[x]$.
We have $p\ x : P\ x$ but $q\ c[x] : P\ d[c[x]]$, so $x \equiv d[c[x]]$ is necessary. This means eta rules for all constructs are needed.

Why eta rules and proof irrelevance ?

Let $A, B : \texttt{Type}$, $c : A \rhd B$, $d : B \rhd A$.
Let $P : A \to \texttt{Prop}$ with $p : \Pi x : A, P\ x$ and $q : \Pi x : B, P\ x$.
Consider: $p\ x =_{(P\ x)} q\ x$ in context $\Gamma \triangleq x : A$.
By $[\![\_]\!]_\Gamma$: $[\![P]\!]_\Gamma \triangleq P$, $[\![p]\!]_\Gamma \triangleq p$ and $[\![q]\!]_\Gamma \triangleq q : \Pi x : B, P\ d[x]$,
hence: $[\![p\ x =_{(P\ x)} q\ x]\!]_\Gamma \triangleq p\ x =_{(P\ x)} q\ c[x]$.
We have $p\ x : P\ x$ but $q\ c[x] : P\ d[c[x]]$, so $x \equiv d[c[x]]$ is necessary. This means eta rules for all constructs are needed.
Now: $c \triangleq \text{elt}\ A\ P\ \cdot\ ?_{P[\cdot/x]} : A \rhd \{\ x : A \mid P\ \}$ and
$d \triangleq \sigma_1 \cdot : \{\ x : A \mid P\ \} \rhd A$, so $d[c[x]] \triangleq x$ but what about $c[d[x]]$?

Why eta rules and proof irrelevance ?

Let $A, B : \texttt{Type}$, $c : A \rhd B$, $d : B \rhd A$.
Let $P : A \to \texttt{Prop}$ with $p : \Pi x : A, P\ x$ and $q : \Pi x : B, P\ x$.
Consider: $p\ x =_{(P\ x)} q\ x$ in context $\Gamma \triangleq x : A$.
By $[\![\_]\!]_\Gamma$: $[\![P]\!]_\Gamma \triangleq P$, $[\![p]\!]_\Gamma \triangleq p$ and $[\![q]\!]_\Gamma \triangleq q : \Pi x : B, P\ d[x]$,
hence: $[\![p\ x =_{(P\ x)} q\ x]\!]_\Gamma \triangleq p\ x =_{(P\ x)} q\ c[x]$.
We have $p\ x : P\ x$ but $q\ c[x] : P\ d[c[x]]$, so $x \equiv d[c[x]]$ is necessary. This means eta rules for all constructs are needed.
Now: $c \triangleq \text{elt}\ A\ P\ \cdot\ ?_{P[\cdot/x]} : A \rhd \{\ x : A \mid P\ \}$ and
$d \triangleq \sigma_1 \cdot : \{\ x : A \mid P\ \} \rhd A$, so $d[c[x]] \triangleq x$ but what about $c[d[x]]$?
$c[d[x]] \triangleq \text{elt}\ A\ P\ (\sigma_1\ x)\ ?_{P[\sigma_1 x/x]} \not\equiv x$ except if PI is included in $\equiv$.

Dependent Inductive coercion rule

$$\frac{\Gamma, (\overrightarrow{x_i : X_i \simeq y_i : Y_i})_0^{j-1} \vdash ?_j : x_j : X_j \simeq y_j : Y_j \quad \forall j \in [0..i]}{\Gamma \vdash c(\overrightarrow{?_i}) : I\ \overrightarrow{x_i} \rhd I\ \overrightarrow{y_i} : s}$$

Splitting nodes

$$
\begin{array}{l}
\texttt{Program Definition}\ \mathit{splitNode}\ (p : v \to \mathit{bool})\ (i : v)\ (n : \mathit{node\ measure}) : \\
\{\ s : \mathit{split}\ (\texttt{fun}\ A \Rightarrow \mathit{option}\ (\mathit{digit}\ A))\ A \mid \texttt{let}\ (l, x, r) := s\ \texttt{in} \\
\quad \texttt{let}\ \mathit{ls} := \mathit{option\_digit\_measure\ measure}\ l\ \texttt{in} \\
\quad \texttt{let}\ \mathit{rs} := \mathit{option\_digit\_measure\ measure}\ r\ \texttt{in} \\
\quad \mathit{node\_measure}\ n = \mathit{ls} \cdot \parallel x \parallel \cdot \mathit{rs} \wedge \\
\quad \mathit{node\_to\_list}\ n = \mathit{option\_digit\_to\_list}\ l \mathbin{+\!\!+} [\mathtt{x}] \mathbin{+\!\!+} \mathit{option\_digit\_to\_list}\ r \wedge \\
\quad (l = \mathit{None} \vee p\ (i \cdot \mathit{ls}) = \mathit{false}) \wedge \\
\quad (r = \mathit{None} \vee p\ (i \cdot \mathit{ls} \cdot \parallel x \parallel) = \mathit{true})\} := \ldots
\end{array}
$$

Instanciation: sequences

The monoid and measure: executable semantics

```
Definition below i := { x : nat | x < i }.
Definition v := { i : nat & (below i → A) }.
Program Definition ε : v := 0 ≺ (fun _ ⇒ !).
Program Definition append (xs ys : v) : v :=
  let (n, fx) := xs in let (m, fy) := ys in
    (n + m) ≺
      (fun i ⇒ if lt_ge_dec i n then fx i else fy (i - n)).
Program Definition seqMonoid := @mkMonoid v ε append _ _ _.
Program Definition measure (x : A) : v := 1 ≺ (fun _ ⇒ x).
```

Specialization

```
Definition seq (x : v) := fingertree seqMonoid measure x.
Program Fixpoint make (i : nat) (v : A) { struct i } : seq (i ≺ (fun _ ⇒ v)).
Program Definition set (i : nat) (m : below i → A) (x : seq (i ≺ m))
  (j : below i) (value : A)
  : seq (i ≺ (fun idx : below i ⇒ if eq_nat_dec idx j then value else m idx)).
Program Definition get (i : nat) (m : below i → A) (x : seq (i ≺ m))
  (j : below i) : { value : A | m j = value }.
```